Toric Varieties as Probability Spaces

based on arXiv:2204.06414
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Outline:

1. Toric varieties and statistical models
2. Positive toric varieties as probability spaces
3. Tropical sampling

0 Discrete statistical models

Definition:

A discrete statistical model with $m+1$ states is a parameterized subset of the probability m-simplex

$$\Delta_m = \left\{ (p_0, \dots, p_m) \;\middle|\; p_i \in (0,1),\ \sum_{i=0}^{m} p_i = 1 \right\}.$$

1 Toric varieties and statistical models

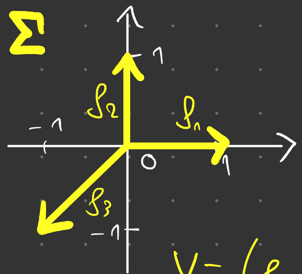
Definition:

An algebraic variety X is toric if it contains a dense algebraic torus whose action on itself extends to X.

Fact:

Normal toric varieties of dimension n are encoded by complete fans in $\mathbb{R}^n$.

Example: Complex projective plane



$V = (\rho_1 | \rho_2 | \rho_3)$

$$X_\Sigma = \mathbb{P}^2_{\mathbb{C}} = (\mathbb{C}^3)^* / \mathbb{C}^* = \mathbb{A}^3_{\mathbb{C}} \setminus V(x_0, x_1, x_2) / \mathbb{G}^1_m$$

homogeneous coordinates $(x_0 : x_1 : x_2)$
"Cox coordinates"

Three affine charts: $\{x_i \neq 0\}$
→ one for each maximal cone

More general: Σ a complete fan in $\mathbb{R}^n$ (e.g. inner normal fan of a polytope P)

- $X_\Sigma = \mathbb{C}^k \setminus V(B) / G$ the toric variety associated to Σ

  (k = #rays; G = characters of the divisor class group of X_Σ)

- $\pi : \mathbb{C}^k \setminus V(B) \longrightarrow \mathbb{C}^k \setminus V(B) / G$

- $\pi(\mathbb{R}^k_{>0}) =: X_{>0}$ the <u>positive part of X_Σ</u>.

Example: **A coin model**

$X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, homogeneous coordinates $(x_0:x_1), (s_0:s_1), (t_0:t_1)$

$\parallel$

X_Σ for Σ the inner normal fan of the cube $[0,1]^3$

$X_{>0} = (0,1)^3$

Model: image of the map

$$X_{>0} \longrightarrow \Delta_m, \quad (x_0,x_1,s_0,s_1,t_0,t_1) \mapsto \left(p_\ell\left(x_0,x_1,s_0,s_1,t_0,t_1\right)\right)_{\ell=0,\dots,m},$$

where

$$p_\ell := \binom{m}{\ell} \frac{x_0}{x_0+x_1} \frac{s_0^\ell s_1^{m-\ell}}{(s_0+s_1)^m} + \binom{m}{\ell} \frac{x_1}{x_0+x_1} \frac{t_0^\ell t_1^{m-\ell}}{(t_0+t_1)^m}.$$

Dehomogenized $(x = x_0,\ x_1 = 1-x,\ s = s_0,\ s_1 = 1-s,\ t = t_0,\ t_1 = 1-t)$:

$$p_\ell = \binom{m}{\ell} x \cdot s^\ell \cdot (1-s)^{m-\ell} + \binom{m}{\ell} (1-x) t^\ell \cdot (1-t)^{m-\ell}.$$

For uniform prior on $(0,1)^3$, $u = (u_0,\dots,u_m)$, the marginal likelihood integral is

$$\int_{X_{>0}} p_0^{u_0} \cdots p_m^{u_m} \, \Omega_X^{\text{unif}}.$$

likelihood function

prior distribution

② $X_{>0}$ as probability space

Ω_X canonical form on X, restriction to $(\mathbb{C}^*)^n \subset X$: $\frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n}$

$\hookrightarrow \Omega_X = \sum_{\substack{I \subset \Sigma(1), \\ |I| = n}} \det(V_I) \cdot \bigwedge_{i \in I} \frac{dx_i}{x_i}$

$(X, X_{>0})$ is a "positive geometry" (Arkani-Hamed / Bai / Lam)

$\frac{f}{g}$ rational function on X, homogeneous in Cox coordinates

$\hookrightarrow f, g \in \mathbb{C}[x_1, \dots, x_k] = \bigoplus_{\gamma \in Cl(X)} S_\gamma$

Consider the integral $\mathcal{I} := \int_{X_{>0}} \frac{f}{g} \Omega_X$. Then $\frac{1}{\mathcal{I}} \cdot \int_{X_{>0}} \frac{f}{g} \Omega_X = 1$.

Then:

- $\frac{1}{\mathcal{I}} \cdot \frac{f}{g} =: d_{f,g}$ is a probability density on $X_{>0}$ w.r.t. Ω_X.
- $\mu_{f,g} := d_{f,g} \cdot \Omega_X$ turns $(X_{>0}, \mu_{f,g})$ into a probability space!

Proposition:

The pullback of $dy_1 \wedge \dots \wedge dy_n$ under the moment map $X_{>0} \longrightarrow P^\circ$ is a positive rational function times the canonical form Ω_X, i.e.,

$$\Omega_X^{\mathrm{unif}} = r(x_1, \dots, x_k) \cdot \Omega_X .$$

Bayesian Integrals on Toric Varieties:

- integrals of the form $\int_{X_{>0}} \frac{f}{g} \Omega_X$ M. Borinsky: Tropical sampling for $X = \mathbb{P}^n$
- sampling algorithm, detour via tropical geometry
- computation of Bayes' factors for model selection
- Julia code provided as Jupyter notebook at

https://mathrepo.mis.mpg.de/BayesianIntegrals

③ Tropical Sampling

The tropical approximation of $f \in \mathbb{C}[x_1, \dots, x_k]$ is

$$f^{tr} : \mathbb{R}^k_{>0} \longrightarrow \mathbb{R}^k, \qquad x \longmapsto \max_{\ell \in supp(f)} x^\ell .$$

$$I^{tr} := \int_{X_{>0}} \frac{f^{tr}}{g^{tr}} \Omega_X, \qquad d^{tr}_{f,g} := \frac{1}{I^{tr}} \frac{f^{tr}}{g^{tr}}, \qquad \mu^{tr}_{f,g} := d^{tr}_{f,g} \cdot \Omega_X$$

$\mathcal{F}$ a simplicial refinement of the normal fan of $N(f \cdot g) = N(f) + N(g)$

K lineality space of $\mathcal{F}$

$\operatorname{Exp}: \mathbb{R}^k / K \longrightarrow X_{>0}, \quad [(y_1, \dots, y_k)] \longmapsto \pi(e^y)$

Sector integrals:

For $\sigma \in \mathcal{F}$, v_f and v_g vertices of corresponding faces of $N(f)$ and $N(g)$,

$$\frac{f^{tr}(x)}{g^{tr}(x)} = x^{-(v_g - v_f)} \quad \forall x \in \mathbb{R}^k \text{ s.t. } \pi(x) \in \operatorname{Exp}(\sigma).$$

$I^{tr}_\sigma := \int_{\operatorname{Exp}(\sigma)} \frac{f^{tr}}{g^{tr}} \Omega_X$ converges and $I^{tr} = \sum_{\sigma \in \mathcal{F}(n)} I^{tr}_\sigma$.

Sampling from $(X_{>0}, d^{tr}_{f,g})$:

Input: $\mathcal{F}(n)$, $v_g - v_f$, $\mathcal{I}^{tr}_{\sigma}$ and $\mathcal{I}^{tr}$

1. Draw an n-dim. σ from $\mathcal{F}(n)$ with probability $\mathcal{I}^{tr}_{\sigma} / \mathcal{I}^{tr}$.
2. Draw a sample q from $[0,1]^n$ using the uniform distribution.
3. Compute $x^{\sigma}(q) \in \mathrm{Exp}(\sigma)$.

Output: $x^{\sigma}(q) \in X_{>0}$, a tropical sample from $X_{>0}$

Then for $h = \frac{f}{g} \cdot \frac{g^{tr}}{f^{tr}}$:

$$I \approx I_N = \frac{\mathcal{I}^{tr}}{N} \sum_{i=1}^{N} h(x^{(i)})$$

i.i.d. tropical samples from $X_{>0}$

Sampling from $(X_{>0}, d_{f,g})$: rejection sampling from $d^{tr}_{f,g} \geq C \cdot d_{f,g}$.

Main references:

- N. Arkani-Hamed, Y. Bai, and T. Lam: Positive geometries and canonical forms, J. High Energy Phys. 11 (2017) 1–124.
- M. Borinsky: Tropical Monte Carlo quadrature for Feynman integrals, Annales de l'Institut Henri Poincaré D, to appear. arXiv:2008.12310.
- M. Borinsky, A.-L. Sattelberger, B. Sturmfels, and S. Telen: Bayesian Integrals on Toric Varieties. arXiv:2204.06414.

Thanks for your attention!