ALGEBRAIC ANALYSIS OF FEYNMAN INTEGRALS

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Vector Spaces of Generalized Euler Integrals + REFs therein

Outline:

1. Algebras of differential & shift operators
2. Mellin transform & Bernstein-Sato ideals
3. Twisted deRham cohomology

0 Motivation

$x = (x_1, \dots, x_n)$

$f = (f_1, \dots, f_\ell) \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^\ell$

$s \in \mathbb{C}^\ell, \; \nu \in \mathbb{C}^n, \; a \in \mathbb{Z}^\ell, \; b \in \mathbb{Z}^n$

Object of study: generalized Euler integrals

$$I_{a,b}(\nu) := \int_\Gamma f(x;c)^{s+a} \, x^{\nu+b} \, \frac{dx}{x}$$

$\frac{dx}{x} = \frac{dx_1}{x_1} \wedge \dots \wedge \frac{dx_n}{x_n}$

Γ: integration contour $\in H_n(X, \omega)$

Example:

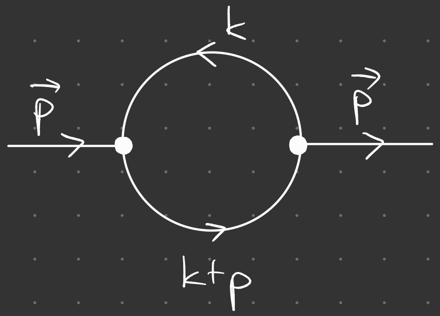
G a Feynman diagram

$f := \mathcal{G} = \mathcal{U} + \mathcal{F}$ the sum of the first and second Symanzik polynomial of G

$a = b = 0, \quad s = -d/2$ $\quad$ d dimension of Minkowski spacetime

Then: $I_{a,b}$ is the Lee-Pomeransky representation of I_G. $\quad$ I_G: Feynman integral of G

Example: Bubble loop with equal masses



G Feynman diagram

$p_1, \dots, p_n$ particles

$k_1, \dots, k_g$ loop momenta

g genus of G

Here: $n=1$, $g=1$.

d dimension of Minkowski spacetime

For each internal edge e: $q_e(k_1, \dots, k_g, p_1, \dots, p_n)$

Via Feynman rules, one associates the Feynman integral

$$I_G = \begin{pmatrix} \text{normalization} \\ \text{factor} \end{pmatrix} \cdot \int_{(\mathbb{R}^{1,d-1})^g} \prod_{\substack{\text{internal} \\ \text{edges } e}} \frac{1}{(-q_e^2 + m_e)^{\nu_e}} \, dk_1^d \wedge \dots \wedge dk_g^d .$$

Further representations of Feynman integrals

To each internal edge of G, associate α_e. "Schwinger parameter"

Let $\mathcal{U}, \mathcal{F} \in \mathbb{C}[\alpha_1, \ldots, \alpha_N]$ be the first and second Symanzik polynomial of G,

and $\mathcal{G} := \mathcal{U} + \mathcal{F}$ their sum.

In our example:
- $\mathcal{U} = \alpha_1 + \alpha_2$
- $\mathcal{F} = (\alpha_1 + \alpha_2)(m^2 \alpha_1 + m^2 \alpha_2 - p^2 \alpha_2) + \alpha_2^2 p^2$

Schwinger representation:

$$I_G = \begin{pmatrix} \text{norm.} \\ \text{factor} \end{pmatrix} \cdot \int_{\alpha_e \geq 0} \left(\prod_e \alpha_e^{\nu_e} \right) \mathcal{U}(\alpha)^{-d/2} e^{-\mathcal{F}(\alpha)/\mathcal{U}(\alpha)} \frac{d\alpha_1}{\alpha_1} \wedge \ldots \wedge \frac{d\alpha_N}{\alpha_N}$$

Lee-Pomeransky representation:

$$I_G = \begin{pmatrix} \text{norm.} \\ \text{factor} \end{pmatrix} \int_{\{\alpha_e \geq 0\}} \mathcal{G}(\alpha)^{-d/2} \prod_e \alpha_e^{\nu_e} \frac{d\alpha_1}{\alpha_1} \wedge \cdots \wedge \frac{d\alpha_N}{\alpha_N}$$

Theorem:

For generic choices each, the following numbers coincide:

- $\left|\chi\big(\underbrace{(\mathbb{C}^*)^n \setminus V(f_1\cdots f_\ell)}_{=:X}\big)\right|$

- $\dim_{\mathbb{C}}\left(H^n_{dR}(X,\omega)\right)$ $\quad \omega := d\log\left(f^s x^\nu\right)$

- $\dim_{\mathbb{C}}\left(\mathrm{span}_{\mathbb{C}}\left\{ \underset{H_n(X,\omega)}{[\Gamma]} \longmapsto \int_\Gamma f^{s+a} x^{\nu+b}\,\frac{dx}{x}\right\}_{a,b\in\mathbb{Z}^\ell\times\mathbb{Z}^n}\right)$

- $\dim_{\underset{c=(c_1,\ldots,c_N)}{\mathbb{C}(c)}}\left(\mathbb{C}(c)\langle\partial_c\rangle \big/ \mathbb{C}(c)\langle\partial_c\rangle\cdot H_A(\kappa)\right) = N!\cdot \mathrm{vol}(A\cup\{0\})$

GKZ system (encoded by integer matrix A + parameter vector $\kappa=(-\nu,s)$)

- $\dim_{\mathbb{C}(s,\nu)}\Big\{\underbrace{\sum_{a,b\in\mathbb{Z}^\ell\times\mathbb{Z}^n}\mathbb{C}(s,\nu)\cdot I_{a,b}}_{=:V_{s,\nu}}\Big\}$

This number is the number of master integrals

① Algebras of differential & shift operators

Definition. $D_{\mathbb{G}_m^n} := \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]\langle \partial_1, \ldots, \partial_n \rangle$

The Weyl algebra $D_n := \mathbb{C}[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n \rangle$ is the free $\mathbb{C}$-algebra generated by $x_1, \ldots, x_n, \partial_1, \ldots, \partial_n$, modulo the following relations: all generators are assumed to commute, except ∂_i, x_i:

D is non-commutative!

$$[\partial_i, x_i] = \partial_i x_i - x_i \partial_i = 1. \qquad \text{(Leibniz' rule)}$$

The Weyl algebra gathers linear differential operators with polynomial coefficients:

$$D_n = \Big\{ \sum_{\substack{k \in \mathbb{N}^n \\ \text{(finite)}}} a_k \partial^k \;\Big|\; a_k \in \mathbb{C}[x_1, \ldots, x_n] \Big\}.$$

Left D-ideals encode systems of linear PDEs.

Definition: The shift algebra S_n is the free $\mathbb{C}$-algebra generated by $\nu_1, \ldots, \nu_n, \sigma_1^{\pm 1}, \ldots, \sigma_n^{\pm 1}$ modulo the following relations: all generators commute, except

$$\sigma_i^{\pm 1} \nu_i = (\nu_i \pm 1)\, \sigma_i^{\pm 1}.$$

$\sigma^a \nu^b = (\nu + a)^b \sigma^a, \; a \in \mathbb{Z}^n, \; b \in \mathbb{N}^n$

② Mellin transform & Bernstein–Sato ideals

Definition:

The Mellin transform of a function $f\colon \mathbb{R}^n_{>0} \to \mathbb{C}$ is

$$\mathcal{M}\{f\}(\nu_1,\dots,\nu_n) = \int_{\mathbb{R}^n_{>0}} f(x)\, x^{\nu-1}\, dx.$$

If f has rapid decay at $0,\infty$: $\mathcal{M}\{f\}$ conv.

It obeys:

i^{th} standard unit vector

$$\mathcal{M}\{x_i f\}(\nu) = \mathcal{M}\{f\}(\nu + e_i) \quad \text{and} \quad \mathcal{M}\{x_i\partial_i \bullet f\} \overset{\text{IBP}}{=} -\nu_i\, \mathcal{M}\{f\}(\nu).$$

Algebraically, the Mellin transform is an isomorphism of $\mathbb{C}$-algebras:

$$\mathbb{C}[x_1^{\pm 1},\dots,x_n^{\pm 1}]\langle\partial_1,\dots,\partial_n\rangle \xrightarrow{\cong} S_n, \quad x_i\partial_i \mapsto -\nu_i, \quad x_i^{\pm 1} \mapsto \sigma_i^{\pm 1}.$$

$\mathcal{D}$ acts on fcts. f $\qquad$ $\mathcal{S}$ acts on Mellin transform: $\sigma_i^{\pm 1} \bullet \mathcal{M}\{f\}(\nu) = \mathcal{M}\{f\}(\nu \pm e_i)$

Observation: $\mathcal{M}\{\cdot\}$ turns $\operatorname{Ann}_{D_{\mathbb{G}_m^n}}(f)$ into $\operatorname{Ann}_{S_n}(\mathcal{M}\{f\})$!

$\Gamma = \mathbb{R}^n_{>0}$, f, s fixed: $\quad I_{0,0}(\nu) = \mathcal{M}\{f^s\}(\nu)$

Definition:

a-Bernstein–Sato ideal ($a \in \mathbb{N}^\ell$)

Let $F = (f_1, \dots, f_\ell) \in \mathbb{C}[x_1, \dots, x_n]^\ell$. The Bernstein–Sato ideal of F is the $\mathbb{C}[s_1, \dots, s_\ell]$-ideal B_F generated by all $p \in \mathbb{C}[s_1, \dots, s_\ell]$ s.t. $\exists P \in D_n[s_1, \dots, s_\ell]$ for which

$$P \bullet \left(f_1^{\overset{s_1+a_1}{s_1+1}} \cdots f_\ell^{\overset{s_\ell+a_\ell}{s_\ell+1}}\right) = p \cdot f_1^{s_1} \cdots f_\ell^{s_\ell}. \qquad (*)$$

For $\ell = 1$: ($F = f$) unique monic generator b_f of B_F, the Bernstein–Sato polynomial of f.

Nota bene:

- $\mathfrak{M}\{P\} \bullet \overbrace{\mathfrak{M}\{f^{s+1}\}}^{I_{1,0}} = b_f(s) \cdot \overbrace{\mathfrak{M}\{f^s\}}^{I_{0,0}}$ lowering in s: $s+1 \longrightarrow s$
- $\underbrace{\mathfrak{M}\{f^{s+1}\}}_{I_{1,0}} = \mathfrak{M}\{f \cdot f^s\} = \underbrace{\mathfrak{M}\{f\}}_{\in S_n} \bullet \underbrace{\mathfrak{M}\{f^s\}}_{I_{0,0}}$ increasing in s: $s \longrightarrow s+1$

$\partial^b \bullet I_{a,0} = I_{a,b}$

$$\Longrightarrow \sum_{(a,b) \in \mathbb{Z}^\ell \times \mathbb{Z}^n} \mathbb{C}(s,\nu) \cdot I_{a,b} = \mathbb{C}(s,\nu) \cdot S_n[s] \bullet I_{0,0}.$$

In particular, it is sufficient to consider shifts in ν!

③ Twisted deRham cohomology

Let $f_1, \ldots, f_\ell \in \mathbb{C}[x_1, \ldots, x_n]$, $X := (\mathbb{C}^*)^n \setminus V(f_1 \cdots f_\ell) = \{x \in (\mathbb{C}^*)^n \mid f_1(x) \cdots f_\ell(x) \neq 0\}$.

$$\omega := \operatorname{dlog}(f^s x^\nu) = \sum_{j=1}^{\ell} s_j \frac{df_j}{f_j} + \sum_{i=1}^{n} \nu_i \frac{dx_i}{x_i} \in \Omega_X^1(X)$$

$$\Omega_X^k(X) = \Big\{ \sum_{i_1 < \cdots < i_k} h_{i_1, \ldots, i_k} \frac{dx_{i_1}}{x_{i_1}} \wedge \cdots \wedge \frac{dx_{i_k}}{x_{i_k}} \;\Big|\; h_{i_1, \ldots, i_k} \in \mathcal{O}_X(X) \Big\}$$ regular k-forms on X

$\mathcal{O}_X(X) = \mathbb{C}[x^{\pm 1}, f^{\pm 1}]$ regular functions on X

Twisted de Rham complex of X:

$$(\Omega_X^\bullet(X), \nabla_\omega): \quad 0 \to \Omega_X^0(X) \xrightarrow{\nabla_\omega^{(0)}} \Omega_X^1(X) \longrightarrow \cdots \xrightarrow{\nabla_\omega^{(n-1)}} \Omega_X^n(X) \longrightarrow 0$$

$\nabla_\omega := d + \omega \wedge$

$$H^k(X, \omega) := H^k(\Omega_X^\bullet(X), \nabla_\omega) = \ker\big(\nabla_\omega^{(k+1)}\big) \big/ \operatorname{im}\big(\nabla_\omega^{(k)}\big)$$ k^{th} twisted cohomology vector space of X

Theorem:

For generic $(s, \nu) \in \mathbb{C}^{\ell+n}$: $H^*(X, \omega) = 0$ for $* \neq n$, $\dim_{\mathbb{C}}(H^n(X, \omega)) = |\chi(X)|$.

Theorem:

homology of X with coefficients in the dual local system of flat sections of $\nabla_\omega = d - \omega\wedge$

$$\langle \cdot , \cdot \rangle : H^n(X,\omega) \times H_n(X,\omega) \longrightarrow \mathbb{C}, \quad \langle [\phi], [\Gamma] \rangle \mapsto \int_\Gamma f^s x^\nu \phi$$

is a perfect pairing.

Hence: $H^n(X,\omega) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{C}}(H_n(X,\omega), \mathbb{C})$ is an isomorphism.

$$\left[f^a x^b \frac{dx}{x} \right] \longmapsto \left(I_{a,b} : \Gamma \mapsto \int_\Gamma f^{s+a} x^{\nu+b} \frac{dx}{x} \right) \quad (**)$$

Moreover: $\displaystyle\sum_{a,b} \overset{\in \mathbb{C}}{C_{a,b}} \cdot I_{a,b} = 0 \iff \sum_{a,b} C_{a,b} f^a x^b \frac{dx}{x} \in \mathrm{im}(\nabla_\omega)$.

IBP relations via $(n-1)$-forms: in purely cohomological terms!

For all $\phi \in \Omega_X^{n-1}(X)$, we get a relation of $I_{a,b}$'s by writing

$$\nabla_\omega(\phi) = \sum_{a,b} C_{a,b}(\phi) \cdot f^a x^b \frac{dx}{x}.$$

Relating IBP and Mellin

Proposition ($\ell=1$):

Let $P \in \mathrm{Ann}_{D_n[s]}(f^s) = \{P \in D_n[s] \mid P \bullet f^s = 0\}$ be of degree at most 1 in the ∂_i, i.e.,

$$P = \sum_{i=1}^{n} p_i(x,s)\,\partial_i + q(x,s) \in D_n[s]$$

for some $p_1, \dots, p_n, q \in \mathbb{C}[x_1, \dots, x_n, s]$.

Then $\mathcal{M}\{P\} \bullet \mathcal{M}\{f^s\} = 0$ and $(**)$ with $\phi = -\sum_{i=1}^{n} s_i\, p_i\, f^{s-1}\, \partial_i \bullet f$ lead to the same linear relation of integrals $I_{a,b}$.

④ Numerical approach ($\ell = 1$):

$f \in \mathbb{C}[x_1, \dots, x_n]$, $\quad \nu \in \mathbb{C}^n$, $s \in \mathbb{C}$

$X = (\mathbb{C}^*)^n \setminus V(f) = \{ x \in (\mathbb{C}^*)^n \mid f(x) \neq 0 \}$, $\quad \chi := |\chi(X)|$

$$\omega = d\log(f^s x^\nu) = s \cdot \frac{df}{f} + \sum_{i=1}^n \nu_i \frac{dx_i}{x_i}$$

Assume a basis $[\Gamma_1], \dots, [\Gamma_\chi]$ of $H_n(X, \omega)$ is given.

Let $(a^{(1)}, b^{(1)}), \dots, (a^{(\chi+1)}, b^{(\chi+1)}) \in \mathbb{N} \times \mathbb{Z}^n$ $\rightsquigarrow$ cocycles $f^{a(j)} x^{b(j)} \frac{dx}{x} \in H^n(X, \omega)$

Proposition:

Define $M \in \mathbb{C}^{\chi \times \chi+1}$ whose $(i,j)^{th}$ entry is $M_{ij} := \langle [f^{a(j)} x^{b(j)} \frac{dx}{x}], [\Gamma_i] \rangle \in \mathbb{C}$.

Then: every $(c_1, \dots, c_{\chi+1}) \in \ker(M)$ gives a linear relation

$$\sum_{j=1}^{\chi+1} c_j I_{a(j), b(j)} = 0. \qquad I_{a(j), b(j)}(\Gamma_i) = \int_{\Gamma_i} f^{s+a(j)} x^{\nu + b(j)} \frac{dx}{x}$$

⑤ $I_{0,0}$ as A-hypergeometric function (for simplicity presented here for $\ell=1$)

Read as function of the coefficients $c_1,\ldots,c_N$ of $f\in\mathbb{C}[x_1^{\pm1},\ldots,x_n^{\pm1}]$, $I_{0,0}(c)=\int_{\mathbb{R}^n_{>0}} f^s x^\nu \frac{dx}{x}$

$N = \#$ monomials in f

is the solution of a GKZ system.

"A-hypergeometric system"

$\kappa=(-\nu,s)^T\in\mathbb{C}^{n+1}$

For $f=\sum_{j=1}^N c_j x^{\nu_j}$, $\nu_j\in\mathbb{Z}^n$, $c_j\in\mathbb{C}$, denote by $A:=\begin{pmatrix}\nu_1 & \cdots & \nu_N\\ 1 & \cdots & 1\end{pmatrix}\in\mathbb{Z}^{(n+1)\times N}$.

Let

- I_A be the toric ideal of A. $I_A := \langle \partial^u-\partial^v \mid u,v\in\mathbb{N}^n,\ u-v\in\ker(A)\rangle \subseteq \mathbb{C}[\partial_1,\ldots,\partial_N]$
- $J_{A,\kappa}$ be the D_N-ideal generated by the entries of $A\cdot\Theta-\kappa$.

  $D_N = \mathbb{C}[c_1,\ldots,c_N]\langle\partial_1,\ldots,\partial_N\rangle$

  $\Theta=(\Theta_1,\ldots,\Theta_N)$ with $\Theta_i=c_i\partial_i$ the i^{th} Euler operator
- $H_A(\kappa) := I_A + J_{A,\kappa}\subseteq D_N$ GKZ system

  Gelfand–Kapranov–Zelevinsky

Then:

$I_{0,0}(c)\in \mathrm{Sol}(H_A(\kappa))$, i.e., $P\bullet I_{0,0}=0$ for all $P\in H_A(\kappa)$.