$D$-Module Techniques for Solving Differential Equations in the Context of Feynman Integrals

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Motivation

Aim: exploit algebraic geometry behind Feynman integrals
  ◦ extraction of properties of Feynman integrals from their PDEs
  ◦ algorithmic computation of series solutions of PDEs by algebraic methods
  ◦ evaluation of Feynman integrals
  ◦ providing a dictionary between algebraic analysis and high energy physics

Outline

1. Algebraic analysis
   \[ D = \mathbb{C}[x_1, \ldots, x_n] \langle \partial_1, \ldots, \partial_n \rangle \]

2. Algebraic computation of solutions
   \[ F_k(x) = x^A \cdot \sum_{p, b \text{ suitable}} c_{pb} x^p \log(x)^b \]

3. Merging \(D\)-module and physics methods
Definition

The **Weyl algebra** is obtained from the free algebra over \( \mathbb{C} \)

\[
D := \mathbb{C}[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n \rangle
\]

by imposing the following relations:

\[
[\partial_i, x_j] = \partial_i x_j - x_j \partial_i = \delta_{ij} \quad \text{for } i, j = 1, \ldots, n.
\]

**From PDEs to** \( D \)-**ideals and vice versa**

- \( D \) gathers linear differential operators with polynomial coefficients

\[
P = \sum_{\alpha, \beta \in \mathbb{N}^n} c_{\alpha, \beta} x^\alpha \partial^\beta, \quad c_{\alpha, \beta} \in \mathbb{C} \quad \hookrightarrow \text{PDE: } P \cdot f(x_1, \ldots, x_n) = 0
\]

**Example:** \( P = \partial^2 - x \in D \) encodes Airy’s equation \( f''(x) - x \cdot f(x) = 0 \).

- **left** \( D \)-ideals encode systems of linear PDEs

operations with \( D \)-ideals: integral transforms, restrictions, push forward, . . .
Holonomic functions

One variable
A function $f(x)$ is **holonomic** if there exists $P \in D$ that annihilates $f$, i.e., $P \cdot f = 0$.

Multivariate case: $f(x_1, \ldots, x_n)$ is holonomic if $\text{Ann}_D(f)$ is a “holonomic” $D$-ideal.

**Examples:** Feynman integrals, hypergeometric, periods, Airy, polylogarithms, . . .

Denote by $R_n = \mathbb{C}(x_1, \ldots, x_n)(\partial_1, \ldots, \partial_n)$ the **rational Weyl algebra**.

**Theorem (Cauchy–Kovalevskaya–Kashiwara)**
Let $I$ be a holonomic $D$-ideal. The $\mathbb{C}$-vector space of holomorphic solutions to $I$ on a simply connected domain in $\mathbb{C}^n$ outside the singular locus of $I$ has finite dimension

$$\text{rank}(I) = \dim_{\mathbb{C}(x_1, \ldots, x_n)} \left( R_n / R_n I \right).$$

⇒ A holonomic function is encoded by finite data!

**Singlarities**

$D$-ideals can be **regular singular** or **irregular singular**.

**Univariate case:** read from growth behavior of general solution near singular points

**Example:** $\diamond \log(x)$ moderate growth at $x = 0$  $\diamond \exp(1/x)$ essential singularity at $x = 0$

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Running example

Variables: \( x_1 = |p_1|^2, \quad x_2 = |p_2|^2, \quad x_3 = |p_1 + p_2|^2. \)

The \( D \)-ideal \( I_3(c_0, c_1, c_2, c_3) \)

Consider \( I_3(c_0, c_1, c_2, c_3) = \langle P_1, P_2, P_3 \rangle \subset D_3 \) arising from \textit{conformal invariance}. dilatations + conformal boosts

\[
\begin{align*}
P_1 &= 4(x_1 \partial_1^2 - x_3 \partial_3^2) + 2(2 + c_0 - 2c_1)\partial_1 - 2(2 + c_0 - 2c_3)\partial_3, \\
P_2 &= 4(x_2 \partial_2^2 - x_3 \partial_3^2) + 2(2 + c_0 - 2c_2)\partial_2 - 2(2 + c_0 - 2c_3)\partial_3, \\
P_3 &= (2c_0 - c_1 - c_2 - c_3) + 2(x_3 \partial_3 + x_2 \partial_2 + x_1 \partial_1).
\end{align*}
\]

Parameters: \( c_0 = d \) spacetime dimension \quad \( c_1, c_2, c_3 \) conformal weights

Choice: \( I_3 := I_3(4, 2, 2, 2) \cong \text{conformal } \phi^4\text{-theory in } 4 \text{ spacetime dimensions} \)

\( I_3 \) is regular singular, \( \text{rank}(I_3) = 4 \)

Remark: The \( D \)-ideal \( I_3 \) is the restriction of a GKZ system.
Solutions to $I_3$

The solution space of $I_3$...

...is spanned by the triangle integral

$$\mathcal{J}_{\text{triangle}}^{d;\nu_1,\nu_2,\nu_3} = \int_{\mathbb{R}^d} \frac{d^d k}{i\pi^d} \frac{1}{(-|k|^2)^{\nu_1} (-|k+p_1|^2)^{\nu_2} (-|k+p_1+p_2|^2)^{\nu_3}}$$

and its analytic continuations. $\text{rank}(I_3) = 4$

Unitary exponents $\nu_1 = \nu_2 = \nu_3 = 1, \ d = 4$:

$$f_1(x_1, x_2, x_3) = \mathcal{J}_{\text{triangle}}^{4;1,1,1}(x_1, x_2, x_3),$$

$$f_2(x_1, x_2, x_3) = \frac{1}{\sqrt{\lambda}} \log \left( \frac{x_1 - x_2 - x_3 - \sqrt{\lambda}}{x_1 - x_2 - x_3 + \sqrt{\lambda}} \right),$$

$$f_3(x_1, x_2, x_3) = \frac{1}{\sqrt{\lambda}} \log \left( \frac{x_2 - x_1 - x_3 - \sqrt{\lambda}}{x_2 - x_1 - x_3 + \sqrt{\lambda}} \right),$$

$$f_4(x_1, x_2, x_3) = \frac{1}{\sqrt{\lambda}},$$

where $\lambda = x_1^2 + x_2^2 + x_3^2 - 2(x_1 x_2 + x_1 x_3 + x_2 x_3)$ is the Källén function.
Initial forms

Principal symbol \((n = 1)\)
\[
in_{(0,1)}(x\partial - x^2) = x\xi
\]
is the part of maximal \((0,1)\)-weight \(\partial \rightsquigarrow \xi\)

Several variables: \(\in_{(0,1)}(x_1\partial_1 + x_2\partial_2 + 1) = x_1\xi_1 + x_2\xi_2\) in general, not a monomial

Algebraically

\- The **characteristic ideal** of a \(D\)-ideal \(I\) is
\[
in_{(0,1)}(I) = \langle \in_{(0,1)}(P) | P \in I \rangle \subset \mathbb{C}[x_1, \ldots, x_n][\xi_1, \ldots, \xi_n].
\]

\- The **characteristic variety** of \(I\) is
\[
\text{Char}(I) = V(\in_{(0,1)}(I)) = \{(x, \xi) | p(x, \xi) = 0 \text{ for all } p \in \in_{(0,1)}(I)\} \subset \mathbb{C}^{2n}.
\]

\- The **singular locus** \(\text{Sing}(I)\) of \(I\) is the vanishing set of the ideal
\[
\left( \in_{(0,1)}(I) : \langle \xi_1, \ldots, \xi_n \rangle^{(\infty)} \right) \cap \mathbb{C}[x_1, \ldots, x_n]. \quad \text{saturation + elimination}
\]

Examples

1. For \(I = \langle x^2\partial + 1 \rangle \subset D\), \(\in_{(0,1)}(I) = \langle x^2\xi \rangle\) and \(\text{Sing}(I) = V(x) = \{0\} \subset \mathbb{C}. \quad \mathbb{C} \cdot \exp(1/x)
\]

2. The characteristic ideal of \(I = \langle x_1\partial_2, x_2\partial_1 \rangle \subset D_2\) is the \(\mathbb{C}[x_1, x_2, \xi_1, \xi_2]-\text{ideal} \langle x_1\xi_2, x_2\xi_1, x_1\xi_1 - x_2\xi_2, x_2\xi_2^2, x_2^2\xi_2 \rangle\) and \(\text{Sing}(I) = V(x_1, x_2) \subset \mathbb{C}^2. \quad \mathbb{C} \cdot 1\)
Gröbner deformations

Weights of the form \((-w, w), w = (w_1, \ldots, w_n) \in \mathbb{R}^n\)

- The \emph{w-weight} of \(c_{\alpha, \beta} x^\alpha \partial^\beta\) is \(-w \cdot \alpha + w \cdot \beta\).
- The \emph{initial form} of \(P = \sum c_{\alpha, \beta} x^\alpha \partial^\beta\) is the subsum of all terms of maximal \(w\)-weight.

Initial and indicial ideal (with respect to \(w\))

- The \emph{initial ideal} of \(I\) is the \(D\)-ideal
  \[\text{in}_w(I) = \langle \text{in}_{(-w, w)}(P) | P \in I \rangle \subset D.\]
- The \emph{indicial ideal} of \(I\) is the \(\mathbb{C}[\theta_1, \ldots, \theta_n]\)-ideal
  \[\text{ind}_w(I) = R_n \cdot \text{in}_{(-w, w)}(I) \cap \mathbb{C}[\theta_1, \ldots, \theta_n].\]
  \(\theta_i = x_i \partial_i\) the \(i\)-th Euler operator

The zeroes of \(\text{ind}_w(I)\) in \(\mathbb{C}^n\) are the \emph{exponents} of \(I\).
The starting monomials of solutions to \(I\) will be of the form \(x^A \log(x)^B\) with \(A \in V(\text{ind}_w(I))\).

Pipeline: from \(I\) to starting terms of series solutions

\[D_n\text{-ideal } I \xrightarrow{\sim} \text{in}_{(-w, w)}(I) \xrightarrow{\sim} \text{ind}_w(I) \subset \mathbb{C}[\theta_1, \ldots, \theta_n] \xrightarrow{\sim} x^A \log(x)^B\]
Canonical series solutions

**Aim:** Solutions to $I$ of the form $F_k(x) = x^A \cdot \sum_{0 \leq p \cdot w \leq k, p \in \mathbb{Z}_*} c_{pb} x^p \log(x)^b$.

**Initial series**

The **$w$-weight** of a monomial $x^A \log(x)^B$ is the real part of $w \cdot A$. The **initial series** $\text{in}_w(f)$ of a function $f = \sum_{A,B} c_{AB} x^A \log(x)^B$ is the subsum of all terms of **minimal** $w$-weight.

**Proposition**

If $I$ is regular holonomic and $w$ a generic weight for $I$, there exist $\text{rank}(I)$ many canonical series solutions of $I$ which lie in the **Nilsson ring** $\mathcal{N}_w(I)$ of $I$ with respect to $w$,

$$\mathcal{N}_w(I) := \mathbb{C}[\mathcal{C}_w(I)^*][x^{e_1}, \ldots, x^{e_r}, \log(x_1), \ldots, \log(x_n)].$$

- $\mathcal{C}_w(I)^*$ the dual cone of the Gröbner cone of $w$
- $\mathcal{C}_w(I)^*_\mathbb{Z} = \mathcal{C}_w(I)^* \cap \mathbb{Z}^n$
- $\{e^1, \ldots, e^r\}$ the exponents of $I$

**Monomial ordering** $\prec_w$ refining $w$-weight: The number of solutions to $I$ with starting monomial of the form $x^A \log(x)^B$ is the multiplicity of $A$ as zero of $\text{ind}_w(I)$.

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Theorem (Saito–Sturmfels–Takayama)

Let \( I \) be a regular holonomic \( \mathbb{Q}[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n \rangle \)-ideal and \( w \in \mathbb{R}^n \) generic for \( I \). Let \( I \) be given by a Gröbner basis for \( w \). There exists an algorithm which computes all terms up to specified \( w \)-weight in the canonical series solutions to \( I \) with respect to \( \prec_w \).

Procedure

Input: A regular holonomic \( D_n \)-ideal \( I \), its small Gröbner fan \( \Sigma \) in \( \mathbb{R}^n \), a weight vector \( w \in \mathbb{R}^n \) that is generic for \( I \), and the desired order \( k \in \mathbb{N} \).

\[ \ldots \text{for each starting monomial } x^A \log(x)^B : \text{solving linear system modulo desired } w \text{-weight for vector spaces of monomials of same } w \text{-weight}. \quad \text{recurrence relations} \]

Output: The canonical series solutions of \( I \) with respect to \( w \), truncated at \( w \)-weight \( k \).
Starting monomials for \( I_3 \)

The singular locus of \( I_3 \) is

\[
\text{Sing}(I_3) = V(x_1x_2x_3 \cdot \lambda) \subset \mathbb{C}^3.
\]

Vanishing locus of the Källén polynomial

\[
\lambda = x_1^2 + x_2^2 + x_3^2 - 2(x_1x_2 + x_1x_3 + x_2x_3)
\]

+ coordinate hyperplanes \( \{ x_i = 0 \} \)

Initial and indicial ideal for \( w = (-1, 0, 1) \in C_1 \)

\[
\begin{align*}
\diamond \text{in}_{(-w,w)}(I_3) &= \langle x_1 \partial_1 + x_2 \partial_2 + x_3 \partial_3 + 1, x_2 \partial_2^2 + \partial_2, x_3 \partial_3^2 + \partial_3 \rangle \subset D_3 \\
\diamond \text{ind}_w(I_3) &= R_3 \cdot \text{in}_{(-w,w)}(I) \cap \mathbb{C}[\theta_1, \theta_2, \theta_3] = \langle \theta_1 + \theta_2 + \theta_3 + 1, \theta_2^2, \theta_3^2 \rangle \subset \mathbb{C}[\theta_1, \theta_2, \theta_3]
\end{align*}
\]

Exponents of \( I \):

\[
V(\text{ind}_w(I_3)) = \{(-1, 0, 0)\}. \quad \cong x_1^{-1}x_2^0x_3^0 = 1/x_1
\]

Change of variables:

\[
y_1 = x_1, \quad y_2 = x_2/x_1, \quad y_3 = x_3/x_1.
\]

Starting monomials of solutions read from primary decomposition of \( \text{ind}_w(I) \)

\[
\diamond 1/y_1 \quad \diamond 1/y_1 \log(y_2) \quad \diamond 1/y_1 \log(y_3) \quad \diamond 1/y_1 \log(y_2) \log(y_3)
\]
Canonical series solutions of $I_3$

Lifting the starting monomials here displayed for $f_1, f_2, f_3$ for $w$-weight 0 to 4

$$\tilde{f}_1(y_2, y_3) = 1 + y_2 + y_3 + y_2^2 + 4y_2y_3 + y_3^2 + 9y_2^2y_3 + y_2^4 + \cdots,$$

$$\tilde{f}_2(y_2, y_3) = \log(y_2) + \log(y_2)y_2 + (2 + \log(y_2))y_3 + \log(y_2)y_2^2 + (4 + 4\log(y_2))y_2y_3$$
$$+ (3 + \log(y_2))y_3^2 + (\log(y_2))y_2^3 + (6 + 9\log(y_2))y_2^2y_3 + \log(y_2)y_2^4 + \cdots,$$

$$\tilde{f}_3(y_2, y_3) = \log(y_3) + (2 + \log(y_3))y_2 + \log(y_3)y_3 + (3 + \log(y_3))y_2^2$$
$$+ (4 + 4\log(y_3))y_2y_3 + \log(y_3)y_3^2 + \left(\frac{11}{3} + \log(y_3)\right)y_2^3$$
$$+ (15 + 9\log(y_3))y_2^2y_3 + \left(\frac{25}{6} + \log(y_3)\right)y_2^4 + \cdots.$$

Then $f_i(x_1, x_2, x_3) = 1/x_1 \cdot \tilde{f}_i(y_2, y_3)$ are canonical series solutions to $I_3$. (truncated)

Implementation in Sage for the bivariate case

Available at: https://mathrepo.mis.mpg.de/DModulesFeynman/
Truncation with respect to $w$-weight

$f(x_1, \ldots, x_n)$ general solution of a regular holonomic $D$-ideal $I$

Capturing the weight vector via an auxiliary variable

Choose a generic weight $w \in \mathbb{R}^n$ for $I$. Set

$$f_w(t, x_1, \ldots, x_n) := f(t^{w_1}x_1, \ldots, t^{w_n}x_n).$$

Merging with canonical series solutions

1. From $I$, derive a Fuchsian system for $f_w(t, x_1, \ldots, x_n)$.
2. Solve the system via the path-ordered exponential formalism.
3. Compute the asymptotic expansion of $f_w(t, x)$ around $t = 0$:

$$f_w(t, x) = \sum_{k \geq 0} \sum_{m=0}^{m_{\text{max}}} c_{k,m}(x) t^k \log(t)^m.$$

   By construction, $c_{k,m}(x)$ has $w$-weight $k$.
4. Truncate the expansion at $t^k$ and evaluate at $t = 1$. Nota bene: $f_w|_{t=1} \equiv f$.

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In a nutshell

1. $D$-ideals encode crucial properties of their solution functions
e.g. Feynman integrals, arbitrary loop order, irrespective of whether polylogarithmic, etc.

2. Algorithmic computation of truncated series solutions by algebraic methods
   no gauge transform required

3. Evaluation of solution functions to desired $w$-weight
   freedom in choosing weight vector $w$

4. Dictionary algebra–physics
   computing series solutions, Pfaffian system vs. Laporta’s algorithm

Thank you for your attention!
The conformal group

\[ z = (z^0, z^1, \ldots, z^{d-1})^\top \] vector of \( d \)-dimensional spacetime coordinates

\[ z_1 \cdot z_2 := z_1^\top \cdot g \cdot z_2 \] \( g = \text{diag}(1, -1, \ldots, -1) \) the metric tensor

\( p_1, \ldots, p_n \) momentum vectors

<table>
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<tr>
<th>Translations</th>
<th>( z \rightarrow z + \epsilon, \text{ } \epsilon \in \mathbb{R}^d )</th>
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<tr>
<td>(Proper) Lorentz transformations</td>
<td>( z \rightarrow \Lambda \cdot z, \text{ } \Lambda \in \text{SO}(1, d - 1) )</td>
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<tr>
<td>Dilatations</td>
<td>( z \rightarrow e^{\omega} \cdot z, \text{ } \omega \in \mathbb{R} )</td>
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<tr>
<td>Conformal boosts</td>
<td>( z \rightarrow \frac{z -</td>
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</table>

Poincaré group symmetry group of Einstein’s theory of special relativity

conformal group Poincaré + dilatations + conformal boosts

Invariance under...

- translations implies momentum conservation
- Lorentz transformation implies dependency on Mandelstam invariants \( p_k \cdot p_\ell \) only

Generators in position space to momentum space via Fourier transform

- dilatations: \( \mathcal{D}_n = -i \sum_{k=1}^n (z_k \cdot \partial z_k + c_k) \)
- conformal boosts: \( \mathcal{K}_n = i \sum_{k=1}^n [z_k^2 \partial z_k - 2 z_k (z_k \cdot \partial z_k) - 2 c_k z_k] \)

Running example: \( n = 3 \), momenta \( p_1, p_2, p_3 \), variables \( x_i = |p_i|^2 \)

- \( P_3 \) stems from \( \mathcal{D}_3 \)
- \( P_1, P_2 \) stem from \( \mathcal{K}_3 \)
Systems in matrix form

- \( I \) a holonomic \( D_n \)-ideal of rank \( m = \text{rank}(I) \), \( f \in \text{Sol}(I) \)
- \( 1, s_2, \ldots, s_m \) a \( \mathbb{C}(x) \)-basis of \( R_n/R_nI \) standard monomials for a Gröbner basis of \( I \)

Pfaffian system

Set \( F = (f, s_2 \cdot f, \ldots, s_m \cdot f)^\top \). There exist \( P_1, \ldots, P_n \in \mathbb{C}(x_1, \ldots, x_n)^{m \times m} \) for which

\[
\partial_i \cdot F = P_i \cdot F.
\]

The matrices \( P_i \) fulfill \( P_i P_j - P_j P_i = \partial_i \cdot P_j - \partial_j \cdot P_i \) for all \( i, j \). Integrability

If all poles are of order at most 1, the system is Fuchsian. To arrive at a Fuchsian form, one might need a gauge transform. Wasow’s method

Construction of a Pfaffian system  IBP reduction with Laporta’s algorithm

- \( \partial^a \) in \( \partial^a \)
- \( a \) in \( \partial^a \)
- \( \partial^a Q_i = 0 \) in \( R_n/R_nI \)
- \( \mathbb{C}(x) \)-basis of \( R_n/R_nI \)

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<td>IBP identities</td>
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<tr>
<td>set of master integrals</td>
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</table>

The SST algorithm

**Input:** A regular holonomic $D_n$-ideal $I$, its small Gröbner fan $\Sigma$ in $\mathbb{R}^n$, a weight vector $w \in \mathbb{R}^n$ that is generic for $I$, and the desired order $k + 1 \in \mathbb{N}$.

1. Determine a Gröbner basis $G = \{g_1, \ldots, g_d\}$ of $I$ with respect to $w$.
2. Write each $g \in G$ as $x^\alpha g = f - h$ with $\alpha \in \mathbb{Z}^n$ such that $f \in K[\theta_1, \ldots, \theta_n]$ and $h \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]\langle \partial_1, \ldots, \partial_n \rangle$ with $\text{ord}(-w, w)(h) < 0$.
3. Compute the indicial ideal $\text{ind}_w(I)$ and its rank($I$) many solutions. They are the form $x^A \log(x)^B$ with $A \in V(\text{ind}_w(I))$. For each starting of these monomials, carry out Step 4.
4. Assume the partial solution
   \[
   F_k(x) = x^A \cdot \sum_{0 \leq p \cdot w \leq k, p \in C^*_w} c_{pb} x^p \log(x)^b .
   \]
   is known. Solve the linear system
   \[
   (f_1, \ldots, f_d) \cdot E_{k+1}(x) = (h_1 - f_1, \ldots, h_d - f_d) \cdot F_k(x) \mod w\text{-weight } k + 2
   \]
   for $E_{k+1} \in \sum_{p \cdot w = k+1, p \in C^*_w} L'_p$ of $w$-weight $k + 1$. Adding $E_{k+1}$ to $F_k$ lifts $F_k$ to $F_{k+1}$.

**Output:** The canonical series solutions of $I$ with respect to $w$, truncated at $w$-weight $k + 1$.

SST algorithm: a hypergeometric example

Consider the $D$-ideal $I$ generated by $P = \theta(\theta - 3) - x(\theta + a)(\theta + b)$.

1. $I$ is holonomic of rank $\text{ord}_{(0,1)}(P) = 2$.
2. Gröbner fan of $I$: two maximal cones $\pm \mathbb{R}_{\geq 0}$.
3. For the weight $w = 1$, $\text{in}_{(-w,w)}(I) = \langle \theta(\theta - 3) \rangle = \text{ind}_w(I)$.
4. Exponents of $I$: $\text{V}(\text{ind}_w(I)) = \{0, 3\}$. Starting monomials $x^0 = 1$ and $x^3$.
5. Choose $x^3$ as starting monomial, $L_p = \mathbb{C} \cdot \{x^{p+3}, x^{p+3} \log(x)\}$. $x^3 \sum_p c_{p,1} x^p + c_{p,2} x^p \log(x)$
6. Write $P = f - h$, where $f = \theta(\theta - 3)$ and $h = x(\theta + a)(\theta + b)$. Action of $\theta$ on $L_p$:
   
   $\theta \cdot x^{p+3} = (p+3)x^{p+3}$ and $\theta \cdot (x^{p+3} \log(x)) = x^{p+3} + (p+3)x^{p+3} \log(x)$.
   
   Thus, the matrix of the operator $\theta$ in the basis $\{x^{p+3}, x^{p+3} \log(x)\}$ is
   
   $$
   \begin{bmatrix}
   p + 3 & 1 \\
   0 & p + 3
   \end{bmatrix}.
   $$

7. Let $c_{p,1}$ and $c_{p,2}$ be the coefficients of $x^{p+3}$ and $x^{p+3} \log(x)$ in the power series expansion. Then we can write our operators as matrices, and our recurrence as

   $$
   \begin{bmatrix}
   p & 1 \\
   0 & p
   \end{bmatrix}
   \begin{bmatrix}
   p + 3 & 1 \\
   0 & p + 3
   \end{bmatrix}
   \begin{bmatrix}
   c_{p,1} \\
   c_{p,2}
   \end{bmatrix} =
   \begin{bmatrix}
   p - a + 2 & 1 \\
   0 & p - a + 2
   \end{bmatrix}
   \begin{bmatrix}
   p - b + 2 & 1 \\
   0 & p - b + 2
   \end{bmatrix}
   \begin{bmatrix}
   c_{p-1,1} \\
   c_{p-1,2}
   \end{bmatrix}
   $$

   with initial values $c_{0,1} = 1$, $c_{0,2} = 0$. Solving the recurrence yields

   $$
   c_{p,1} = 0 \quad \text{and} \quad c_{p,2} = \frac{(a+3)_p (b+3)_p}{(1)_p (4)_p}.
   $$