$D$-Module Techniques for Solving Differential Equations

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Algebraic analysis in particle physics

Scattering processes of elementary particles

Aim: exploit algebraic geometry behind Feynman integrals

1. extraction of properties of Feynman integrals from their annihilating $D$-ideal
2. algorithmic computation of series solutions of PDEs by algebraic methods
3. simplification of considered systems of PDEs
4. providing a dictionary between algebraic analysis and physics

Holonomic functions

One variable
A function $f(x)$ is **holonomic** if there exists $P \in D$ that annihilates $f$, i.e., $P \cdot f = 0$.

Multivariate case: $f(x_1, \ldots, x_n)$ is holonomic if $\text{Ann}_D(f)$ is a holonomic $D$-ideal.

**Examples:** Feynman integrals, hypergeometric, periods, Airy, polylogarithms, . . .

Denote by $R_n = \mathbb{C}(x_1, \ldots, x_n)\langle \partial_1, \ldots, \partial_n \rangle$ the rational Weyl algebra.

Theorem (Cauchy–Kovalevskaya–Kashiwara)
Let $I$ be a holonomic $D$-ideal. The $\mathbb{C}$-vector space of holomorphic solutions to $I$ on a simply connected domain in $\mathbb{C}^n$ outside the singular locus of $I$ has finite dimension

$$\text{rank}(I) = \dim_{\mathbb{C}(x_1, \ldots, x_n)} (R_n/R_nI).$$

⇒ A holonomic function is encoded by finite data!

Utility of holonomicity

- volume computations
- symbolic manipulations
- maximum likelihood estimation
- numerical evaluation

Running example

Variables: \( x_1 = |p_1|^2 \), \( x_2 = |p_2|^2 \), \( x_3 = |p_1 + p_2|^2 \).

The \( D \)-ideal \( I_3(c_0, c_1, c_2, c_3) \)

Consider \( I_3(c_0, c_1, c_2, c_3) = \langle P_1, P_2, P_3 \rangle \subset D_3 \) arising from \textit{conformal invariance}.

dilatations + conformal boosts

\[
\begin{align*}
P_1 &= 4(x_1 \partial_1^2 - x_3 \partial_3^2) + 2(2 + c_0 - 2c_1)\partial_1 - 2(2 + c_0 - 2c_3)\partial_3, \\
P_2 &= 4(x_2 \partial_2^2 - x_3 \partial_3^2) + 2(2 + c_0 - 2c_2)\partial_2 - 2(2 + c_0 - 2c_3)\partial_3, \\
P_3 &= (2c_0 - c_1 - c_2 - c_3) + 2(x_3 \partial_3 + x_2 \partial_2 + x_1 \partial_1).
\end{align*}
\]

Parameters: \( c_0 = d \) \textit{spacetime dimension} \hspace{1cm} \( c_1, c_2, c_3 \) \textit{conformal weights}

Choice: \( I_3 := I_3(4, 2, 2, 2) \equiv \text{conformal } \phi^4\text{-theory in 4 spacetime dimensions} \)

\( I_3 \) is regular singular, \( \operatorname{rank}(I_3) = 4 \)

Remark: The \( D \)-ideal \( I_3 \) is the restriction of a GKZ system.

Solutions to $l_3$

The solution space of $l_3$... 

... is spanned by the triangle integral

$$\int_{d; \nu_1, \nu_2, \nu_3} \frac{d^d k}{i \pi^2} \frac{1}{(-|k|^2)^{\nu_1} (-|k+p_1|^2)^{\nu_2} (-|k+p_1+p_2|^2)^{\nu_3}}$$

and its analytic continuations.  \( \text{rank}(l_3) = 4 \)

Unitary exponents $\nu_1 = \nu_2 = \nu_3 = 1$, $d = 4$:

$$f_1(x_1, x_2, x_3) = \int_{4; 1, 1, 1}^{\text{triangle}} (x_1, x_2, x_3),$$

$$f_2(x_1, x_2, x_3) = \frac{1}{\sqrt{\lambda}} \log \left( \frac{x_1 - x_2 - x_3 - \sqrt{\lambda}}{x_1 - x_2 - x_3 + \sqrt{\lambda}} \right),$$

$$f_3(x_1, x_2, x_3) = \frac{1}{\sqrt{\lambda}} \log \left( \frac{x_2 - x_1 - x_3 - \sqrt{\lambda}}{x_2 - x_1 - x_3 + \sqrt{\lambda}} \right),$$

$$f_4(x_1, x_2, x_3) = \frac{1}{\sqrt{\lambda}} ,$$

where $\lambda = x_1^2 + x_2^2 + x_3^2 - 2(x_1 x_2 + x_1 x_3 + x_2 x_3)$ is the Källén function.
Initial forms

Principal symbol \((n = 1)\)

\[
in_{(0,1)}(x\partial - x^2) = x\xi
\]
is the part of maximal \((0, 1)\)-weight \(\partial \rightsquigarrow \xi\)

Several variables: \(\in_{(0,1)}(x_1\partial_1 + x_2\partial_2 + 1) = x_1\xi_1 + x_2\xi_2\)
in general, not a monomial

Algebraically

\(\checkmark\) The **characteristic ideal** of a \(D\)-ideal \(I\) is

\[\in_{(0,1)}(I) = \langle \in_{(0,1)}(P) \mid P \in I \rangle \subset \mathbb{C}[x_1, \ldots, x_n][\xi_1, \ldots, \xi_n].\]

\(\checkmark\) The **characteristic variety** of \(I\) is

\[\text{Char}(I) = V(\in_{(0,1)}(I)) = \{(x, \xi) \mid p(x, \xi) = 0 \text{ for all } p \in \in_{(0,1)}(I)\} \subset \mathbb{C}^{2n}.\]

\(\checkmark\) The **singular locus** \(\text{Sing}(I)\) of \(I\) is the vanishing set of the ideal

\[\left( \in_{(0,1)}(I) : \langle \xi_1, \ldots, \xi_n \rangle^{(\infty)} \right) \cap \mathbb{C}[x_1, \ldots, x_n].\]
saturation + elimination

Examples

1. For \(I = \langle x^2\partial + 1 \rangle \subset D\), \(\in_{(0,1)}(I) = \langle x^2\xi \rangle\) and \(\text{Sing}(I) = V(x) = \{0\}. \quad \mathbb{C} \cdot \exp(1/x)\)

2. The characteristic ideal of \(I = \langle x_1\partial_2, x_2\partial_1 \rangle \subset D_2\) is the \(\mathbb{C}[x_1, x_2, \xi_1, \xi_2]-\text{ideal}\)

\[\langle x_1\xi_2, x_2\xi_1, x_1\xi_1 - x_2\xi_2, x_2\xi_2^2, x_2^2\xi_2 \rangle\]
and \(\text{Sing}(I) = V(x_1, x_2) \subset \mathbb{C}^2. \quad \mathbb{C} \cdot 1\)
Gröbner deformations

Weights of the form \((-w, w)\) \(w \in \mathbb{R}^n\)

- The **w-weight** of \(c_{\alpha, \beta} x^\alpha \partial^\beta\) is \(-w \cdot \alpha + w \cdot \beta\).
- The **initial form** of \(P = \sum c_{\alpha, \beta} x^\alpha \partial^\beta\) is the subsum of all terms of maximal \(w\)-weight.

Initial and indicial ideal (with respect to \(w\))

- The **initial ideal** of \(I\) is the \(D\)-ideal
  \[\text{in}_w(I) = \langle \text{in}_{(-w, w)}(P) | P \in I \rangle \subset D\].
- The **indicial ideal** of \(I\) is the \(\mathbb{C}[\theta_1, \ldots, \theta_n]\)-ideal
  \[\text{ind}_w(I) = R_n \cdot \text{in}_{(-w, w)}(I) \cap \mathbb{C}[\theta_1, \ldots, \theta_n]\].

The zeroes of \(\text{ind}_w(I)\) in \(\mathbb{C}^n\) are the **exponents** of \(I\).
The starting monomials of solutions to \(I\) will be of the form \(x^A \log(x)^B\) with \(A \in V(\text{ind}_w(I))\).

Pipeline: from \(I\) to starting terms of series solutions

\[
\begin{align*}
D_n\text{-ideal } I & \xrightarrow{w \in \mathbb{R}^n} \text{in}_{(-w, w)}(I) & \xrightarrow{} \text{ind}_w(I) \subset \mathbb{C}[\theta_1, \ldots, \theta_n] & \xrightarrow{V(\text{ind}_w(I))} x^A \log(x)^B
\end{align*}
\]
Aim: Solutions to $I$ of the form

$$F_k(x) = x^A \cdot \sum_{0 \leq p \cdot w \leq k, p \in \mathbb{C}_*^*} c_{pb} x^p \log(x)^b.$$  

Initial series

The *$w$-weight* of a monomial $x^A \log(x)^B$ is the real part of $w \cdot A$. The **initial series** $in_w(f)$ of a function $f = \sum_{A,B} c_{AB} x^A \log(x)^B$ is the subsum of all terms of minimal $w$-weight.

Proposition

If $I$ is regular holonomic and $w$ a generic weight for $I$, there exist $\text{rank}(I)$ many canonical series solutions of $I$ which lie in the **Nilsson ring** $N_w(I)$ of $I$ with respect to $w$,

$$N_w(I) := \mathbb{C}[[C_w(I)_{\mathbb{Z}}^*]][x^{e_1}, \ldots, x^{e_r}, \log(x_1), \ldots, \log(x_n)].$$

- $C_w(I)^*$ the dual cone of the Gröbner cone of $w$
- $C_w(I)_{\mathbb{Z}}^* = C_w(I)^* \cap \mathbb{Z}^n$
- $\{e_1, \ldots, e_r\}$ the exponents of $I$

**Monomial ordering** $\prec_w$ refining $w$-weight: The number of solutions to $I$ with starting monomial of the form $x^A \log(x)^B$ is the multiplicity of $A$ as zero of $\text{ind}_w(I)$.

The SST algorithm

Theorem (Saito–Sturmfels–Takayama)
Let $I$ be a regular holonomic $\mathbb{Q}[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n \rangle$-ideal and $w \in \mathbb{R}^n$ generic for $I$. Let $I$ be given by a Gröbner basis for $w$. There exists an algorithm which computes all terms up to specified $w$-weight in the canonical series solutions to $I$ with respect to $\prec_w$.

Procedure

**Input:** A regular holonomic $D_n$-ideal $I$, its small Gröbner fan $\Sigma$ in $\mathbb{R}^n$, a weight vector $w \in \mathbb{R}^n$ that is generic for $I$, and the desired order $k \in \mathbb{N}$.

...for each starting monomial $x^A \log(x)^B$: solving linear system modulo desired $w$-weight for vector spaces of monomials of same $w$-weight. *recurrence relations*

**Output:** The canonical series solutions of $I$ with respect to $w$, truncated at $w$-weight $k$.

Starting monomials for $I_3$

The **singular locus** of $I_3$ is

$$\text{Sing} (I_3) = V (x_1 x_2 x_3 \cdot \lambda) \subset \mathbb{C}^3.$$ 

Vanishing locus of the Källén polynomial

$$\lambda = x_1^2 + x_2^2 + x_3^2 - 2(x_1 x_2 + x_1 x_3 + x_2 x_3)$$

+ coordinate hyperplanes $\{ x_i = 0 \}$

Initial and indicial ideal for $w = (-1, 0, 1) \in \mathbb{C}_1$

- $\text{in}_{(-w,w)}(I_3) = \langle x_1 \partial_1 + x_2 \partial_2 + x_3 \partial_3 + 1, x_2 \partial_2^2 + \partial_2, x_3 \partial_3^2 + \partial_3 \rangle \subset D_3$
- $\text{ind}_w(I_3) = R_3 \cdot \text{in}_{(-w,w)}(I) \cap \mathbb{C}[\theta_1, \theta_2, \theta_3] = \langle \theta_1 + \theta_2 + \theta_3 + 1, \theta_2^2, \theta_3^2 \rangle \subset \mathbb{C}[\theta_1, \theta_2, \theta_3]$

Exponents of $I$: $V(\text{ind}_w(I_3)) = \{ (-1, 0, 0) \} \quad \cong x_1^{-1} x_2^0 x_3^0 = 1/x_1$

Change of variables: $y_1 = x_1$, $y_2 = x_2/x_1$, $y_3 = x_3/x_1$.

Starting monomials of solutions read from primary decomposition of $\text{ind}_w(I)$

- $1/y_1$
- $1/y_1 \log(y_2)$
- $1/y_1 \log(y_3)$
- $1/y_1 \log(y_2) \log(y_3)$
Lifting the starting monomials here displayed for \( f_1, f_2, f_3 \) for \( w \)-weight 0 to 4

\[
\begin{align*}
\tilde{f}_1(y_2, y_3) &= 1 + y_2 + y_3 + y_2^2 + 4y_2y_3 + y_3^2 + y_2^3 + 9y_2^2y_3 + y_2^4 + \cdots , \\
\tilde{f}_2(y_2, y_3) &= \log(y_2) + \log(y_2)y_2 + (2 + \log(y_2))y_3 + \log(y_2)y_2^2 + (4 + 4 \log(y_2))y_2y_3 \\
&\quad + (3 + \log(y_2))y_3^2 + (\log(y_2))y_2^3 + (6 + 9 \log(y_2))y_2^2y_3 + \log(y_2)y_2^4 + \cdots , \\
\tilde{f}_3(y_2, y_3) &= \log(y_3) + (2 + \log(y_3))y_2 + \log(y_3)y_3 + (3 + \log(y_3))y_2^2 \\
&\quad + (4 + 4 \log(y_3))y_2y_3 + \log(y_3)y_3^2 + \left( \frac{11}{3} + \log(y_3) \right) y_2^3 \\
&\quad + (15 + 9 \log(y_3))y_2^2y_3 + \left( \frac{25}{6} + \log(y_3) \right) y_2^4 + \cdots .
\end{align*}
\]

Then \( f_i(x_1, x_2, x_3) = 1/x_1 \cdot \tilde{f}_i(y_2, y_3) \) are canonical series solutions to \( I_3 \). (truncated)

Implementation in Sage for the bivariate case

Available at: https://mathrepo.mis.mpg.de/DModulesFeynman/
Solving a four-loop ladder diagram

Variables:

\[ y_1 = |p_4|^2, \quad y_2 = \frac{|p_1+p_2|^2 - |p_4|^2}{|p_4|^2}, \]
\[ y_3 = \frac{|p_4|^2 - |p_1+p_2|^2 - |p_2+p_3|^2}{|p_4|^2}. \]

\( J^{\text{ladder}} \) only depends on \( y_2 \) and \( y_3 \)

Steps:

1. Compute \( P_1, P_2 \in \text{Ann}_{D_2}(J^{\text{ladder}}) \) Gröbner basis of the \( D_2 \)-ideal \( \langle P_1, P_2 \rangle \) hard to compute
2. Gröbner basis in \( R_2 = \mathbb{C}(y_2, y_3)\langle \partial y_2, \partial y_3 \rangle \): \( \{ G_1, G_2 \} \).
3. For \( I = \langle G_1, G_2 \rangle \), its Weyl closure \( W(I) = R_2 I \cap D_2 \) equals \( W(\langle P_1, P_2 \rangle) \). same solution space!
4. Exploiting homogeneity, the system reduces to one operator in \( y = y_2/y_3 \):

\[ \theta^4_y + y\theta^2_y(\theta_y + 1)^2 \]

Solutions: weight \( w = 1 \)

\[ f_1(y) = 1, \quad f_2(y) = \log(y), \quad f_3(y) = \log(y)^2 + 2 \sum_{p=1}^{\infty} \frac{(-y)^p}{p^2} = \log(y)^2 + 2 \text{Li}_2(-y), \]
\[ f_4(y) = \log(y)^3 - 12 \sum_{p=1}^{\infty} \frac{(-y)^p}{p^3} + 6 \log(y) \sum_{p=1}^{\infty} \frac{(-y)^p}{p^2} = \log(y)^3 - 12 \text{Li}_3(-y) + 6 \log(y)\text{Li}_2(-y). \]
In a nutshell

1. **algorithmic** computation of truncated **series solutions** by algebraic methods
   no gauge transform required

2. **simplification** of system of PDEs
   Gröbner bases in rational Weyl algebra, Weyl closure

3. **evaluation** of solution functions to desired $w$-weight
   freedom in choosing weight vector $w$

4. **dictionary** algebra–physics
   computing series solutions, Pfaffian system vs. Laporta’s algorithm

**Muchas gracias por su atención!**

Four-loop ladder integral

The four-loop “ladder” integral with one off-shell leg is $|p_4|^2 \neq 0$

$$J_d^{\text{ladder}} = \int \left( \prod_{j=1}^{4} \frac{d^d k_j}{i \pi^{d/2}} \right) \frac{(|p_1 + p_2|^2)^{2(6-d)} |p_2 + p_3|^2}{|k_1|^2 |k_1 + p_1|^2 |k_1 + p_1 + p_2|^2 |k_2|^2 |k_2 - k_1|^2 |k_2 + p_1 + p_2|^2} \times$$

$$\frac{1}{|k_3|^2 |k_3 - k_2|^2 |k_3 + p_1 + p_2|^2 |k_4|^2 |k_4 - k_3|^2 |k_4 + p_1 + p_2|^2 |k_4 - p_4|^2},$$

where the integration domain is the Minkowski spacetime.

Let $P_1, P_2 \in \text{Ann}_{D_2}(J_d^{\text{ladder}})$ as in [(C.3), HPSZ24]. A Gröbner basis of $\langle P_1, P_2 \rangle$ in $R_2$ is:

$$G_1 = y_2 \partial_{y_2} + y_3 \partial_{y_3},$$

$$G_2 = y_3^2 (y_2 + y_3) \partial_{y_3}^4 + y_3 (4y_2 + 6y_3) \partial_{y_3}^3 + (2y_2 + 7y_3) \partial_{y_3}^2 + \partial_{y_3}. $$

$W(\langle G_1, G_2 \rangle) = RI \cap D = W(\langle P_1, P_2 \rangle)$ as $D_2$-ideals.
The conformal group

\[ z = \begin{pmatrix} z^0, z^1, \ldots, z^{d-1} \end{pmatrix}^\top \]

vector of \( d \)-dimensional spacetime coordinates

\[ z_1 \cdot z_2 := z_1^\top g \cdot z_2 \]
momentum vectors

\[ g = \text{diag}(1, -1, \ldots, -1) \] the metric tensor

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<th>Translations</th>
<th>( z \mapsto z + \epsilon, \ \epsilon \in \mathbb{R}^d )</th>
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<td>(Proper) Lorentz transformations</td>
<td>( z \mapsto \Lambda \cdot z, \ \Lambda \in \text{SO}(1, d-1) )</td>
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<tr>
<td>Dilatations</td>
<td>( z \mapsto e^{\omega} z, \ \omega \in \mathbb{R} )</td>
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<td>Conformal boosts</td>
<td>( z \mapsto \frac{z -</td>
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Poincaré group symmetry group of Einstein’s theory of special relativity

conformal group Poincaré + dilatations + conformal boosts

Invariance under...

- translations implies momentum conservation
- Lorentz transformation implies dependency on Mandelstam invariants \( p_k \cdot p_\ell \) only

Generators in position space to momentum space via Fourier transform

- dilatations:
  \[ \mathcal{D}_n = -i \sum_{k=1}^n (z_k \cdot \partial_{z_k} + c_k) \]
- conformal boosts:
  \[ \mathcal{K}_n = i \sum_{k=1}^n \left[ |z_k|^2 \partial_{z_k} - 2 z_k (z_k \cdot \partial_{z_k}) - 2 c_k z_k \right] \]

Running example: \( n = 3 \), momenta \( p_1, p_2, p_3 \), variables \( x_i = |p_i|^2 \)

- \( P_3 \) stems from \( \mathcal{D}_3 \)
- \( P_1, P_2 \) stem from \( \mathcal{K}_3 \)
Systems in matrix form

◊ $I$ a holonomic $D_n$-ideal of rank $m = \text{rank}(I)$, $f \in \text{Sol}(I)$
◊ $1, s_2, \ldots, s_m$ a $\mathbb{C}(x)$-basis of $R_n/R_nI$ e.g., standard monomials for a Gröbner basis of $R_nI$

Pfaffian system

Set $F = (f, s_2 \cdot f, \ldots, s_m \cdot f)^\top$. There exist $P_1, \ldots, P_n \in \mathbb{C}(x_1, \ldots, x_n)^{m \times m}$ for which

\[
\partial_i \cdot F = P_i \cdot F.
\]

The matrices $P_i$ fulfill $P_i P_j - P_j P_i = \partial_i \cdot P_j - \partial_j \cdot P_i$ for all $i, j$. 

If all poles are of order at most 1, the system is Fuchsian. To arrive at a Fuchsian form, one might need a gauge transform. 

Wasow’s method

Construction of a Pfaffian system IBP reduction with Laporta’s algorithm

\[
\partial^a
\]

Feynman integrals

$\partial^a$ a in $\partial^a$

propagator powers

$\partial^a Q_i = 0$ in $R_n/R_nI$

IBP identities

$\mathbb{C}(x)$-basis of $R_n/R_nI$

set of master integrals

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The SST algorithm

**Input:** A regular holonomic $D_n$-ideal $I$, its small Gröbner fan $\Sigma$ in $\mathbb{R}^n$, a weight vector $w \in \mathbb{R}^n$ that is generic for $I$, and the desired order $k + 1 \in \mathbb{N}$.

1. Determine a Gröbner basis $G = \{g_1, \ldots, g_d\}$ of $I$ with respect to $w$.
2. Write each $g \in G$ as $x^\alpha g = f - h$ with $\alpha \in \mathbb{Z}^n$ such that $f \in \mathbb{K}[\theta_1, \ldots, \theta_n]$ and $h \in \mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]\langle \partial_1, \ldots, \partial_n \rangle$ with $\text{ord}(-w, w)(h) < 0$.
3. Compute the indicial ideal $\text{ind}_w(I)$ and its rank$(I)$ many solutions. They are the form $x^A \log(x)^B$ with $A \in V(\text{ind}_w(I))$. For each starting of these monomials, carry out Step 4.
4. Assume the partial solution

$$F_k(x) = x^A \cdot \sum_{0 \leq p \cdot w \leq k, p \in C_w^*} c_{pb} x^p \log(x)^b.$$  

is known. Solve the linear system

$$(f_1, \ldots, f_d) \cdot E_{k+1}(x) = (h_1 - f_1, \ldots, h_d - f_d) \cdot F_k(x) \mod w\text{-weight } k + 2$$

for $E_{k+1} \in \sum_{p \cdot w = k+1, p \in C_w^*} L'_p$ of $w$-weight $k + 1$. Adding $E_{k+1}$ to $F_k$ lifts $F_k$ to $F_{k+1}$.

$L'_p$ the subspace of $L_p = x^A \sum_{0 \leq b_i \leq \text{rank}(I)} \mathbb{K} \cdot x^p \log(x)^b$ spanned by monomials $\notin \text{Start}<_w (I)$

**Output:** The canonical series solutions of $I$ with respect to $w$, truncated at $w$-weight $k + 1$.

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SST algorithm: a hypergeometric example

Consider the $D$-ideal $I$ generated by $P = \theta(\theta - 3) - x(\theta + a)(\theta + b)$.

1. $I$ is holonomic of rank $\text{ord}_{(0,1)}(P) = 2$.
2. Gröbner fan of $I$: two maximal cones $\pm \mathbb{R}_{\geq 0}$.
3. For the weight $w = 1$, $\text{in}_{(-w,w)}(I) = \langle \theta(\theta - 3) \rangle = \text{ind}_w(I)$.
4. Exponents of $I$: $V(\text{ind}_w(I)) = \{0, 3\}$. Starting monomials $x^0 = 1$ and $x^3$.
5. Choose $x^3$ as starting monomial, $L_p = \mathbb{C} \cdot \{x^{p+3}, x^{p+3} \log(x)\}$. $x^3 \sum_p c_{p,1}x^p + c_{p,2}x^p \log(x)$.
6. Write $P = f - h$, where $f = \theta(\theta - 3)$ and $h = x(\theta + a)(\theta + b)$. Action of $\theta$ on $L_p$:
   $$\theta \cdot x^{p+3} = (p + 3)x^{p+3} \quad \text{and} \quad \theta \cdot (x^{p+3} \log(x)) = x^{p+3} + (p + 3)x^{p+3} \log(x).$$
   Thus, the matrix of the operator $\theta$ in the basis $\{x^{p+3}, x^{p+3} \log(x)\}$ is
   $$\begin{bmatrix} p + 3 & 1 \\ 0 & p + 3 \end{bmatrix}.$$
7. Let $c_{p,1}$ and $c_{p,2}$ be the coefficients of $x^{p+3}$ and $x^{p+3} \log(x)$ in the power series expansion. Then we can write our operators as matrices, and our recurrence as
   $$\begin{bmatrix} p & 1 \\ 0 & p \end{bmatrix} \begin{bmatrix} p + 3 & 1 \\ 0 & p + 3 \end{bmatrix} \begin{bmatrix} c_{p,1} \\ c_{p,2} \end{bmatrix} = \begin{bmatrix} p - a + 2 & 1 \\ 0 & p - a + 2 \end{bmatrix} \begin{bmatrix} p - b + 2 & 1 \\ 0 & p - b + 2 \end{bmatrix} \begin{bmatrix} c_{p-1,1} \\ c_{p-1,2} \end{bmatrix}.$$
   With initial values $c_{0,1} = 1$, $c_{0,2} = 0$. Solving the recurrence yields
   $$c_{p,1} = 0 \quad \text{and} \quad c_{p,2} = \frac{(a+3)_p(b+3)_p}{(1)_p(4)_p}.$$
Truncation with respect to \( w \)-weight

\[ f(x_1, \ldots, x_n) \]  

general solution of a regular holonomic \( D \)-ideal \( I \)

Capturing the weight vector via an auxiliary variable

Choose a generic weight \( w \in \mathbb{R}^n \) for \( I \). Set

\[ f_w(t, x_1, \ldots, x_n) := f(t^{w_1}x_1, \ldots, t^{w_n}x_n). \]

Merging with canonical series solutions

1. From \( I \), derive a **Fuchsian system** for \( f_w(t, x_1, \ldots, x_n) \).
2. Solve the system via the path-ordered exponential formalism.
3. Compute the asymptotic expansion of \( f_w(t, x) \) around \( t = 0 \):

\[
 f_w(t, x) = \sum_{k \geq 0} \sum_{m=0}^{m_{\text{max}}} c_{k,m}(x) t^k \log(t)^m. 
\]

   By construction, \( c_{k,m}(x) \) has \( w \)-weight \( k \).
4. Truncate the expansion at \( t^k \) and evaluate at \( t = 1 \).  
   **Nota bene:** \( f_w|_{t=1} \equiv f \).

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