Algebraic Aspects of Linear PDEs

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Abstract

These are the lecture notes for a course taught by the author at the school "Amplitools: Mathematical Methods for Particles Physics, Gravitation, and Cosmology" in Domodossola, July 14–18, 2025. The target group consists both of theoretical physicists and mathematicians. The lecture series treats some algebraic aspects of homogeneous, linear partial differential equations (PDEs) with polynomial coefficients, with an emphasis on computations. Algebraically, such PDEs are encoded as elements of the Weyl algebra, denoted D, which is a non-commutative $\mathbb{C}$-algebra. Systems of such PDEs correspond to left modules over D. These modules encode crucial information about their solution functions—such as their singularities, growth behavior, and the number of initial conditions that is required to uniquely encode the function. The mathematical field of algebraic analysis contributed deep foundational insights to the theory of linear PDEs, but it is also useful in applications. For instance, generalized Euler integrals are well studied from the point of view of algebraic analysis. These functions are solutions to D-ideals with a strong combinatorial flavor, namely GKZ (a.k.a. "A-hypergeometric") systems. Feynman integrals are closely tied to them, namely by restricting to the geometric subspace which is of physical interest. Via Gröbner basis techniques in the Weyl algebra, one can modify D-ideals symbolically. This facilitates, for instance, a systematic representation of systems of PDEs in matrix form as well as the computation of series solutions to regular holonomic D-ideals.

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Introduction

Algebraic analysis is a mathematical field which investigates linear ordinary and partial differential equations (ODEs and PDEs) with polynomial coefficients via algebraic methods by encoding them as modules over the Weyl algebra, which is denoted D. This theory not only provides foundational insights into the theory of linear PDEs, but turns out to be useful in applications. This lecture series focuses on the computational aspects of holonomic D-modules and their holomorphic solutions: holonomic functions, which were first studied algorithmically by D. Zeilberger [60] in the context of proving combinatorial identities.

This introduction provides some background and overview of the theory. In full generality, algebraic analysis investigates sheaves of $\mathcal{D}_X$-modules on a complex manifold or a smooth algebraic variety X, where $\mathcal{D}_X$ is the sheaf of differential operators on X. This abstract setting enables the formulation of deep classification results, such as the Riemann–Hilbert correspondence. The latter gives a positive answer to Hilbert's 21st problem in a generalized setup—which has a negative answer in its original formulation. It was phrased in the setup of one complex variable and asked the following: *Given singular points $p_1, \ldots, p_k$ in the complex plane and a representation of the fundamental group of $\mathbb{C}\backslash\{p_1, \ldots, p_k\}$, does there always exist a linear ODE of the Fuchsian class whose solutions have the prescribed singularity and monodromy data?* The (regular) Riemann–Hilbert correspondence is an equivalence of categories which allows to replace a regular holonomic $\mathcal{D}$-module by its topological counterpart, namely its derived solution complex, which is a constructible complex of $\mathbb{C}_X$-vector spaces, cf. [32]. This equivalence was proven by M. Kashiwara and Z. Mebkhout independently. It restricts to an equivalence of categories between integrable (also called "flat") connections on vector bundles with at most regular singularities along a divisor $D \subset X$ and the category of local systems on $X\backslash D$, which are a generalization of representations of the fundamental group. This version of the Riemann–Hilbert correspondence is attributed to P. Deligne. In the presence of irregular singularities, where solution functions can have exponential growth, Stokes data need to be taken into account in order to capture the change of the asymptotic behavior of solutions in dependence on the considered sector around the considered singularity. One of the first occurrences of this phenomenon was for Airy's functions in the investigation of supernumerary rainbows. Airy's functions are solutions to a second-order ODE with an irregular singularity at infinity. In modern language, some Stokes phenomena are described purely topologically in [10, 16]. A Riemann–Hilbert correspondence for non-necessarily regular holonomic $\mathcal{D}$-modules—based on the theory of ind-sheaves—was established by A. D'Agnolo and M. Kashiwara in [18].

The toolkit of $\mathcal{D}$-modules turns out to be useful and practicable in applications as well, where it mainly enters via the Weyl algebra, denoted D_n (or just D if the number of variables is clear from the context). It is the non-commutative ring of global sections $D = \mathcal{D}_X(X)$ for $X = \mathbb{A}^n_{\mathbb{C}}$ the affine n-space over the complex numbers. Modules over D then correspond precisely to sheaves of modules over $\mathcal{D}_{\mathbb{A}^n_{\mathbb{C}}}$ that are quasi-coherent as $\mathcal{O}_X$-modules, see [32, Proposition 1.4.4]. This lecture series focuses on the Weyl algebra; familiarity with sheaf theory is not required. In particular, we mainly deal with D-modules of the form D/I with $I \subset D$ being a left ideal in D. Solution functions to holonomic (also called "maximally overdetermined") systems $I \subset D$ are *holonomic functions*. One can compute with holonomic

D-modules and their solution functions using Gröbner bases in non-commutative rings of differential operators. Weyl algebras are a special case of the broader class of Ore algebras. Those are, for instance, implemented in the package `ore_algebra` [35] by M. Kauers in Sage. A Gröbner basis theory that is custom-tailored to Weyl algebras is elaborated in [50]. Via Gröbner deformations, one can also compute series solutions to regular holonomic D-ideals, generalizing the Frobenius method for 2nd order ODEs to an arbitrary order and number of variables as in [50, Section 2.6].

In Mathematica [34], one can compute symbolically with holonomic functions with the help of the package `HolonomicFunctions` [37] of C. Koutschan. Via the holonomic gradient method and descent of [53], one can evaluate and minimize (real-valued) holonomic functions, making use of the knowledge of an annihilating D-ideal. Another application of D-modules in algebraic geometry are volume computations for compact semi-algebraic sets: exploiting that such volume functions are periods in the sense of M. Kontsevich and D. Zagier—and as such solutions to a Picard–Fuchs operator—they can be evaluated to arbitrary precision [39].

One immediate occurrence of D-modules in particle physics and theoretical cosmology arises via generalized Euler integrals [44]. These are well understood from the perspective of algebraic analysis: they are solutions to GKZ [23] (a.k.a. "A-hypergeometric") systems, for a survey of which I'd like to refer to [48]. These integrals are closely bond to Feynman integrals as well as to correlation functions in cosmological toy models [3, 21]. Feynman integrals [59], in their Lee–Pomeransky representation, are restrictions of GKZ systems and in this sense are inherently tied to A-hypergeometric functions [12, 19]. However, it is a computational challenge to determine the D-ideals restricted to relevant kinematic subspaces of interest explicitly. It is an active area of research to construct differential equations behind Feynman integrals and correlation functions, such as addressed in recent work [5, 7, 46].

In the study of Feynman integrals, algebraic geometry naturally enters in various way. To a Feynman diagram G, one associates the graph polynomial $\mathcal{G}_G = \mathcal{U}_G + \mathcal{F}_G$, namely the sum of the 1st and 2nd Symanzik polynomials. It is a polynomial in the Schwinger parameters of the Feynman graph. The complement of its algebraic variety in the algebraic torus, $X = (\mathbb{C}^*)^n \backslash V(\mathcal{G}_G)$, is a very affine variety [33] whose signed topological Euler characteristic equals the number of master integrals. The Feynman integral itself can be obtained as the pairing of a twisted cycle and cocyle of X. All these different perspectives are presented and linked in [1], which in particular spells out the respective genericity assumptions that are required for the validity of the equalities stated. Algebraic geometry for scattering amplitudes is also utilized in the flourishing field of positive geometry [47, 22], which is currently being established. Among other aims, one here seeks to represent scattering amplitudes as volumes of geometric objects, obtained by integrating against the "canonical form"—a specific rational differential form with a recursive pole structure. Among them, the amplituhedron as introduced by N. Arkani-Hamed and J. Trnka [4]—with variations for different loop levels—is currently most prominent. It has a strong combinatorial flavor and is obtained as a projection of the Plücker non-negative part of a Grassmannian variety. Its canonical form computes amplitudes in $\mathcal{N} = 4$ supersymmetric Yang–Mills in the planar limit. Other physical theories are about to be addressed. It is a current undertaking of positive geometry to formalize these statements rigorously in mathematical terms. For mathematically-aimed studies and reviews, I'd like to refer to [11, 14, 40, 54] and the references therein.

Each section of these notes approximately corresponds to a one-hour lecture. Parts of this manuscript closely follow the lecture notes of the course "Applied Algebraic Analysis" which were developed and taught by the author at KTH and Stockholm University in the fall term 2023 and taught again in a similar format at MPI-MiS Leipzig in winter 2024 under the title "Introduction to Algebraic Analysis."

Outline. Section 1 introduces the Weyl algebra as well as basic concepts from the theory of D-modules. Section 2 presents the characteristic variety and the singular locus of D-ideals. Section 3 treats holonomicity and closure properties of the class of holonomic functions. Section 4 addresses Gröbner bases in Weyl algebras as well as their computation. Section 5 explains how to write systems of (higher-order) PDEs in (first-order) matrix form via Gröbner bases. Section 6 was not taught during the school, but provides additional material; it visits integral transform as well as restriction and integration D-ideals.

1 The (rational) Weyl algebra

Algebraically, linear, homogeneous differential equations with polynomials coefficients are encoded as elements of the Weyl algebra, denoted D. This section introduces this non-commutative ring and discusses first basic properties of ideals in and modules over D.

1.1 Encoding linear PDEs algebraically

Definition 1.1. The (n-th) *Weyl algebra*, denoted D_n or just D if the number of variables is clear from the context, is the algebra obtained from the free algebra over $\mathbb{C}$ generated by variables $x_1, \ldots, x_n$ and partial derivatives $\partial_1, \ldots, \partial_n$

$$D \coloneqq \mathbb{C}[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n \rangle$$

by imposing the following relations: all generators are assumed to commute, except ∂_i and x_i. Their *commutator* $[\partial_i, x_i] \coloneqq \partial_i x_i - x_i \partial_i$ fulfills

$$[\partial_i, x_i] \,=\, 1 \quad \text{for} \;\; i = 1, \ldots, n. \tag{1.1}$$

This encodes precisely Leibniz's rule for taking the derivative of the product of functions: applying (1.1) to a function f of x yields

$$\frac{\partial\,(xf)}{\partial x} - x\frac{\partial f}{\partial x} \,=\, xf' + f - xf' \,=\, 1 \cdot f\,.$$

Exercise 1.2 (Commutators [49, 1.2.4])**.** Prove that, in the Weyl algebra,

(1) $[\partial_i^\ell, x_i] = \ell \partial_i^{\ell-1}$, (2) $[\partial_i, x_i^k] = k x_i^{k-1}$,

(3) $[\partial_i^\ell, x_i^k] = \sum_{j\geq 1} \frac{k(k-1)\cdots(k-j+1)\cdot \ell(\ell-1)\cdots(\ell-j+1)}{j!} x_i^{k-j} \partial_i^{\ell-j}$,

where, by convention, negative powers are 0. ◇

The Weyl algebra gathers differential operators on $\mathbb{C}^n$.[1] Its elements are linear differential operators with polynomial coefficients, i.e., as a set,

$$D \;=\; \left\{ \sum_{\alpha,\beta\in\mathbb{N}^n} c_{\alpha,\beta} x^\alpha \partial^\beta \,|\, c_{\alpha,\beta}\in\mathbb{C},\ \text{only finitely many } c_{\alpha,\beta} \text{ non-zero} \right\},$$

where multi-index notation is used, i.e., $x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ and $\partial^\beta = \partial_1^{\beta_1}\cdots\partial_n^{\beta_n}$.

We denote the action of a differential operator $P\in D$ on a function f by a bullet, i.e., $\partial_i \bullet f = \partial f/\partial x_i$, and so on. In order to stress that a function f depends on variables $x_1,\ldots,x_n$, we sometimes write $f(x_1,\ldots,x_n)$ for the function.

Definition 1.3. The *order* of a differential operator $P = \sum_{\alpha,\beta\in\mathbb{N}} c_{\alpha,\beta}x^\alpha\partial^\beta \in D_1$ is

$$\operatorname{ord}(P) \;\coloneqq\; \max\left\{\beta \,|\, \exists\, \alpha \text{ s.t. } c_{\alpha,\beta}\neq 0\right\},$$

i.e., the highest derivative that occurs in the associated linear ODE, $P\bullet f = 0$.

Example 1.4 (Airy's equation)**.** Airy's equation is the linear, second-order ODE

$$f''(x) - x\cdot f(x) \;=\; 0\,. \tag{1.2}$$

Its solutions describe particles that are confined within a triangular potential well [55]. Moreover, it is the standard example to demonstrate Stokes' phenomenon—a wall-crossing phenomenon for the asymptotic behavior of the solution functions to (1.2), cf. [58]. To the ODE in (1.2), we associate the differential operator $P_{\text{Airy}} = \partial^2 - x$. Its $\mathbb{C}$-vector space of holomorphic solutions is spanned by Airy's functions of first and second kind, Ai and Bi:

$$\begin{aligned}\operatorname{Ai}(x) \;&=\; \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3}+xt\right)\,\mathrm{d}t\,,\\ \operatorname{Bi}(x) \;&=\; \frac{1}{\pi}\int_0^\infty \left[\exp\left(-\frac{t^3}{3}+xt\right) + \sin\left(\frac{t^3}{3}+xt\right)\right]\mathrm{d}t\,.\end{aligned}$$

Algebraically, Airy's function Ai is encoded by the differential operator P_{Airy} together with the following two initial conditions, which pick out Ai from the two-dimensional solution space of (1.2):

$$f(0) \;=\; \frac{1}{3^{2/3}\Gamma\left(\frac{2}{3}\right)} \quad \text{and} \quad f'(0) \;=\; -\frac{1}{3^{1/3}\Gamma\left(\frac{1}{3}\right)}\,,$$

where $\Gamma(\cdot)$ denotes the gamma function, $\Gamma(\nu) = \int_0^\infty e^{-x}x^{\nu-1}\,\mathrm{d}x$. ◇

Definition 1.5. The operator $\theta_i \coloneqq x_i\partial_i \in D_n$ is called *i-th Euler operator*.

[1] To be precise, we are on affine n-space over $\mathbb{C}$, i.e., the affine scheme $\mathbb{A}^n_{\mathbb{C}} = \operatorname{Spec}(\mathbb{C}[x_1,\ldots,x_n])$. The closed points of $\mathbb{A}^n_{\mathbb{C}}$ are $\mathbb{A}^n_{\mathbb{C}}(\mathbb{C}) = \mathbb{C}^n$.

Theorem 1.6 (Converse of Euler's homogeneous function theorem)**.** *If $f(x_1,\ldots,x_n)$ is annihilated by $\theta_1+\cdots+\theta_n-k$, where $k\in\mathbb{Z}$, then f is (positively) homogeneous of degree k, i.e.:*

$$f(\lambda x_1,\ldots,\lambda x_n) \,=\, \lambda^k\cdot f(x_1,\ldots,x_n) \quad \textit{for all} \;\; \lambda>0\,.$$

Proof. Set $g(\lambda)=f(\lambda x_1,\ldots,\lambda x_n)$. Via the chain rule, differentiating g yields

$$\begin{aligned} g'(\lambda) \,&=\, x_1\frac{\partial f}{\partial x_1}(\lambda x_1,\ldots,\lambda x_n)+\cdots+x_n\frac{\partial f}{\partial x_n}(\lambda x_1,\ldots,\lambda x_n)\\ &=\, \frac{1}{\lambda}\cdot\big(\lambda x_1\frac{\partial f}{\partial x_1}(\lambda x_1,\ldots,\lambda x_n)+\cdots+\lambda x_n\frac{\partial f}{\partial x_n}(\lambda x_1,\ldots,\lambda x_n)\big). \end{aligned}$$

Since $(\theta_1+\cdots+\theta_n)\bullet f = k\cdot f$, we conclude that $g'(\lambda)=\frac{k}{\lambda}\cdot g(\lambda)$ and hence $\frac{g'(\lambda)}{g(\lambda)}=\frac{k}{\lambda}$. Integrating both sides of this equation yields that $\ln|g(\lambda)|=\ln(\lambda^k)+\ln(C)$ for all $\lambda>0$ and thus $g(\lambda)=C\lambda^k$. Plugging in $\lambda=1$ yields that $C=g(1)$ and hence $g(\lambda)=\lambda^k\cdot g(1)$. Written out, this yields $f(\lambda x_1,\ldots,\lambda x_n)=\lambda^k\cdot f(x_1,\ldots,x_n)$ for all $\lambda>0$. □

The rings $\mathbb{C}[\partial_1,\ldots,\partial_n]$ and $\mathbb{C}[\theta_1,\ldots,\theta_n]$ are *commutative* subrings of the Weyl algebra. The first denotes linear PDEs with *constant* coefficients. Note that for $z=1/x$, $x\partial_x=-z\partial_z$. Hence, for elements of the $\mathbb{C}[\theta_1,\ldots,\theta_n]$, it is particularly easy to switch from 0 to ∞ and vice versa. Further frequently used rings of differential operators are

$$D_{\mathbb{G}_m^n} \,\coloneqq\, \mathbb{C}[x_1^{\pm1},\ldots,x_n^{\pm1}]\langle\partial_1,\ldots,\partial_n\rangle$$

with coefficients in Laurent polynomials $\mathbb{C}[x_1^{\pm1},\ldots,x_n^{\pm1}]=\mathbb{C}[x_1,x_1^{-1},\ldots,x_n,x_n^{-1}]$, which are exactly the global functions on the algebraic n-torus $(\mathbb{C}^*)^n$,[2] and the *rational Weyl algebra*

$$R_n \,\coloneqq\, \mathbb{C}(x_1,\ldots,x_n)\langle\partial_1,\ldots,\partial_n\rangle$$

with coefficients in the field of rational functions

$$\mathbb{C}(x_1,\ldots,x_n) \,=\, \left\{\frac{p}{q}\,\middle|\; p,q\in\mathbb{C}[x_1,\ldots,x_n],\, q\neq0\right\}.$$

Since the Weyl algebra is non-commutative, we have to distinguish between left and right D-ideals. If not explicitly stated otherwise, we always mean *left* D-ideals, since those correspond to systems of linear PDEs: if $P\bullet f=0$ for some $P\in D$, then also $QP\bullet f=0$ for any $Q\in D$. We will denote by $\langle P_1,\ldots,P_k\rangle$ the left D-ideal generated by $P_1,\ldots,P_k\in D$, and sometimes by DP if $k=1$. The Weyl algebra is simple as a ring, i.e., it does not contain any proper two-sided ideal.

One important example of D-ideals are A-hypergeometric systems—also called "GKZ systems," named after Gelfand, Kapranov, and Zelevinsky [23]. They are determined by an integer matrix $A\in\mathbb{Z}^{n\times k}$ and a parameter vector $\kappa\in\mathbb{C}^n$. To define them, we consider the Weyl algebra $D_A=\mathbb{C}[c_\alpha\,|\,\alpha\in A]\langle\partial_\alpha\,|\,\alpha\in A\rangle$ whose variables are indexed by the columns of A.

[2] To be precise, we are on the algebraic n-torus $\mathbb{G}_m^n=\operatorname{Spec}(\mathbb{C}[x_1^{\pm1},\ldots,x_n^{\pm1}])$. The complex-valued closed points of the algebraic n-torus are $\mathbb{G}_m^n(\mathbb{C})=(\mathbb{C}^*)^n$, and $D_{\mathbb{G}_m^n}=\mathcal{D}_{\mathbb{G}_m^n}(\mathbb{G}_m^n)$.

Definition 1.7. The *toric ideal* associated to A is the binomial ideal

$$I_A \coloneqq \left\langle \partial^u - \partial^v \,|\, u - v \in \ker(A),\ u, v \in \mathbb{N}^A \right\rangle \subset \mathbb{C}[\partial_\alpha \,|\, \alpha \in A]\,, \tag{1.3}$$

where $u = (u_\alpha)_{\alpha \in A} \in \mathbb{N}^A$, $\partial^u = \prod_{\alpha \in A} \partial_\alpha^{u_\alpha}$, and similarly for v.

Let $J_{A,\kappa}$ be the ideal generated by the entries of $A\theta - \kappa$ where $\theta \coloneqq (\theta_\alpha)_{\alpha \in A}$ and $\theta_\alpha = c_\alpha \partial_\alpha$.

Definition 1.8. Let $A \in \mathbb{Z}^{n \times k}$ and $\kappa \in \mathbb{C}^n$. The *A-hypergeometric system* (or *GKZ system*) of A, κ is the D_A-ideal $H_A(\kappa) \coloneqq I_A + J_{A,\kappa}$.

Sometimes, one assumes that the all-one vector is contained in the row space of A. This implies that $H_A(\kappa)$ is "regular holonomic," see [50, Theorem 2.4.9]. This property of the D-ideal imposes on its solutions functions that hey have a mild growth behavior near singularities, namely that they are of moderate growth, see [57, Section 5.1].

Example 1.9 ([1, Example 4.2])**.** Let $\kappa = (-\nu_1, -\nu_2, s) \in \mathbb{C}^3$. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 & 3 & 3 \\ 2 & 3 & 1 & 3 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \in \mathbb{Z}^{3 \times 6}\,.$$

Using the `Macaulay2` [26] package `Dmodules` [42], one computes that the toric ideal I_A is generated by 9 binomials, namely

$$\begin{aligned} I_A \;=\; \big\langle\, &\partial_2\partial_5 - \partial_1\partial_6,\, \partial_3\partial_4 - \partial_1\partial_6,\, \partial_4\partial_5^2 - \partial_3\partial_6^2,\, \partial_1\partial_5^2 - \partial_3^2\partial_6,\, \partial_4^2\partial_5 - \partial_2\partial_6^2, \\ &\partial_1\partial_4\partial_5 - \partial_2\partial_3\partial_6,\, \partial_1\partial_4^2 - \partial_2^2\partial_6,\, \partial_2\partial_3^2 - \partial_1^2\partial_5, \partial_2^2\partial_3 - \partial_1^2\partial_4 \,\big\rangle. \end{aligned} \tag{1.4}$$

In the notation of (1.3), the first generator in (1.4) corresponds to the tuple of vectors $u = (0,1,0,0,1,0)$ and $v = (1,0,0,0,0,1)$. The third generator to $u = (0,0,0,1,2,0)$ and $v = (0,0,1,0,0,2)$, and so on. The ideal $J_{A,\kappa}$ is generated by the 3 operators

$$\theta_1+\theta_2+2\theta_3+2\theta_4+3\theta_5+3\theta_6+\nu_1,\ 2\theta_1+3\theta_2+\theta_3+3\theta_4+\theta_5+2\theta_6+\nu_2,\ \theta_1+\theta_2+\theta_3+\theta_4+\theta_5+\theta_6-s.$$

Together, these 12 operators generate $H_A(\kappa) \subset D_6$. ◇

Exercise 1.10. Compute an A-hypergeometric system of your choice using `Macaulay2`.[3] ◇

In the rational Weyl algebra R_1, every left ideal is principal, i.e., can be generated by a single operator. This does not hold true in D_1, but we have the following.

Theorem 1.11 (Stafford)**.** *For every D_n-ideal I, there exist $P, Q \in D_n$ such that $I = \langle P, Q \rangle$.*

An algorithmic proof of the theorem can be found in [41].

Exercise 1.12. Compute two operators that generate the D_4-ideal $\langle \partial_1, \partial_2, \partial_3, \partial_4 \rangle$, for instance by running the the lines below in `Macaulay2`, using the package `Dmodules.m2` [42].

```
loadPackage "Dmodules"
D = QQ[x1,x2,x3,x4,d1,d2,d3,d4,WeylAlgebra=>{x1=>d1,x2=>d2,x3=>d3,x4=>d4}];
I = ideal(d1,d2,d3,d4)
stafford I
```

As pointed out in the documentation, the current implementation of the command `stafford` guarantees the in- and output ideals to be equal only in the rational Weyl algebra R_n. ◇

[3] An online version of `Macaulay2` is available at the following link: https://macaulay2.com/TryItOut/

1.2 D-modules

We now turn to modules over the Weyl algebra.

Definition 1.13. A *D-module* is a left D-module, i.e., an abelian group M together with a left action of the Weyl algebra

$$\bullet\colon\ D \times M \longrightarrow M, \quad (P,m) \mapsto P \bullet m$$

obeying the usual compatibility conditions, i.e., for all $P, Q \in D$ and $m, n \in M$:

(i) $(P \cdot Q) \bullet m \ = \ P \bullet (Q \bullet m)$,

(ii) $(P + Q) \bullet m \ = \ P \bullet m + Q \bullet m$,

(iii) $P \bullet (m + n) \ = \ P \bullet m + P \bullet n$,

(iv) and $1 \bullet m \ = \ m$.

We write $\operatorname{Mod}(D)$ for the category of left D-modules: its objects are left modules over the Weyl algebra, and its morphisms are left D_n-linear maps of D-modules. To a D-ideal I, one associates the D-module $D/I \in \operatorname{Mod}(D)$. In this sense, D-modules are generalizations of systems of linear PDEs.

More generally, consider a system of linear PDEs in ℓ unknown functions $f_1, \ldots, f_\ell$,

$$\sum_{j=1}^{\ell} P_{ij} \bullet u_j \ = \ 0, \qquad i = 1, \ldots, k, \tag{1.5}$$

for some $P_{ij} \in D_n$. To the system (1.5), one associates a D-module M by requiring that it fits into the exact sequence

$$D^k \xrightarrow{\varphi} D^\ell \longrightarrow M \longrightarrow 0$$

of left D-modules, where the first arrow is

$$\varphi(Q_1, \ldots, Q_k) \ = \ \Big(\sum_{i=1}^{k} Q_i P_{i1}, \sum_{i=1}^{k} Q_i P_{i2}, \ldots, \sum_{i=1}^{k} Q_i P_{i\ell} \Big) \,.$$

Then the space of holomorphic solutions to (1.5) is isomorphic to $\operatorname{Hom}_D(M, \mathcal{O})$.

In this course, we focus on D-modules of the form D/I. In fact, by a theorem of Stafford [41], every "holonomic" D-module is of that form. In other words, every holonomic D-module is cyclic, i.e., it is generated by a single element. We will learn about the precise meaning of holonomicity in Section 2.

Example 1.14 (Example 1.9 revisited)**.** The D-module associated to a hypergeometric system $H_A(\kappa)$ is typically denoted by $M_A(\kappa) = D_A/H_A(\kappa) \in \operatorname{Mod}(D_A)$. ⋄

Exercise 1.15 ($n = 1$)**.** Determine for which $a, b \in \mathbb{C}$ there there exists a non-trivial left D-linear morphism between the D-modules $M_a \coloneqq D/(x\partial - a)$ and $M_b \coloneqq D/(x\partial - b)$. ⋄

Exercise 1.16. Let $M \in \operatorname{Mod}(D)$ such that M is finite-dimensional as a $\mathbb{C}$-vector space. Prove that $M = 0$. ⋄

Definition 1.17. Let $M \in \mathrm{Mod}(D)$ and $m \in M$. The *annihilator* of m is the D-ideal

$$\mathrm{Ann}_D(m) \coloneqq \{P \in D \,|\, P \bullet m = 0\}\,. \tag{1.6}$$

Further typical examples of D-modules of a different kind are function spaces, e.g.,

- holomorphic functions $\mathcal{O}^{\mathrm{an}}$,[4]
- rational functions $\mathbb{C}(x)$,
- convergent $\mathbb{C}\{x\}$ or formal $\mathbb{C}[\![x]\!]$ power series,
- convergent $\mathbb{C}\{\{x\}\} = \mathbb{C}\{x\}[x^{-1}]$ or formal $\mathbb{C}(\!(x)\!) = \mathbb{C}[\![x]\!][x^{-1}]$ Laurent series,
- or (complex-valued) Schwartz distributions, $(C_c^\infty)'$.

For $\mathcal{F} \in \mathrm{Mod}(D)$ a function space, the annihilator (1.6) of a function $f \in \mathcal{F}$ has a direct interpretation, namely as the linear PDEs that f fulfills. We will learn in Section 5 how to express these systems of (higher-order) PDEs in (first-order) matrix form by Gröbner basis computations with an annihilating D-ideal of the function.

2 Singular locus

When studying PDEs, one is course interested in its solutions. We are (mainly) going to be interested in holomorphic, i.e., complex analytic solution functions.

2.1 Solution space

Definition 2.1. Let I be a D-ideal and $M \in \mathrm{Mod}(D)$. The *solution space of I in M* is the $\mathbb{C}$-vector space

$$\mathrm{Sol}_M(I) \;\coloneqq\; \{m \in M \mid P \bullet m = 0 \;\text{ for all }\; P \in I\}\,. \tag{2.1}$$

Also holomorphic functions $\mathcal{O}^{\mathrm{an}}$ on (subsets of) $\mathbb{C}^n$ are a D-module; they carry a natural left action of the Weyl algebra. Unless stated otherwise, we will deal with holomorphic solutions on a suitable open domain $U \subset \mathbb{C}^n$, and we drop the subscript in (2.1) to mean such solutions, i.e., we denote

$$\mathrm{Sol}(I) \;=\; \mathrm{Sol}_{\mathcal{O}^{\mathrm{an}}}(I) \;=\; \{f \in \mathcal{O}^{\mathrm{an}} \,|\, P \bullet f = 0 \;\text{ for all }\; P \in I\}$$

for suitable $U \subset \mathbb{C}^n$. The solution space of a D-module can be recovered completely algebraically. For two D-modules $M, N \in \mathrm{Mod}(D)$, denote by $\mathrm{Hom}_D(M,N)$ the space of morphisms of left D-modules. Now let $I = DP$ for some $P \in D$. The $\mathbb{C}$-vector spaces

$$\mathrm{Hom}_D\left(D/DP, \mathcal{O}^{\mathrm{an}}\right) \;\cong\; \{f \in \mathcal{O}^{\mathrm{an}} \mid P \bullet f = 0\} \tag{2.2}$$

[4] Of course, holomorphic functions are always to be thought of *locally*, i.e., one means $\mathcal{O}^{\mathrm{an}}(U)$ for some appropriate domain $U \subset \mathbb{C}^n$.

are isomorphic. The isomorphism (2.2) is obtained by combining the isomorphisms

$$\mathrm{Hom}_D(D/DP, \mathcal{O}^{\mathrm{an}}) \;\cong\; \{\varphi \in \mathrm{Hom}_D(D, \mathcal{O}^{\mathrm{an}}) \,|\, \varphi(P) = 0\}$$

and

$$\mathrm{Hom}_D(D, \mathcal{O}^{\mathrm{an}}) \;\cong\; \mathcal{O}^{\mathrm{an}}, \quad \varphi \mapsto \varphi(1)\,,$$

cf. [32, p. 2] for more details. This implies that the space of holomorphic solutions $\mathrm{Sol}(I)$ of I is isomorphic to $\mathrm{Hom}_D\left(D/I, \mathcal{O}^{\mathrm{an}}\right)$. Because of the identity in (2.2), the functor $\mathcal{H}om_D(\cdot, \mathcal{O}^{\mathrm{an}})$ is called "solution functor."[5]

Remark 2.2. In (2.2), on may equivalently directly move on to the anlytified setup. We will denote by $(\cdot)^{\mathrm{an}}$ the analytification, so that $D^{\mathrm{an}} = \mathbb{C}\{x\}\langle\partial\rangle$. By some yoga with forgetful functors and flat morphisms, one can argue that

$$\mathrm{Hom}_D\left(D/I, \mathcal{O}^{\mathrm{an}}\right) \;\cong\; \mathrm{Hom}_{D^{\mathrm{an}}}\left(D^{\mathrm{an}}/D^{\mathrm{an}}I, \mathcal{O}^{\mathrm{an}}\right) \tag{2.3}$$

as $\mathbb{C}$-vector spaces. ◇

Remark 2.3. In order not to neglect non-algebraic solutions like the exponential function or the logarithm, it is important to consider D-linear morphisms to $\mathcal{O}^{\mathrm{an}}$—and not only to $\mathcal{O}$ (with $\mathcal{O}_{\mathbb{C}^n}(\mathbb{C}^n) = \mathbb{C}[x_1, \ldots, x_n]$). ◇

Remark 2.4. One may look for solutions of a D-module M in any D-module N: by what was argued above, the vector space $\mathrm{Hom}_{D^{\mathrm{an}}}(M^{\mathrm{an}}, N^{\mathrm{an}})$ encodes the *solutions of M in N*. ◇

Theorem 2.5 (Cauchy–Kovalevskaya–Kashiwara)**.** *Let I be D-ideal and let U be an open subset of $\mathbb{C}^n \setminus \mathrm{Sing}(I)$ that is simply connected. If I is holonomic, then the $\mathbb{C}$-vector space of holomorphic functions on U that are solutions to the system of PDEs encoded by I has dimension equal to* $\mathrm{rank}(I)$*, i.e.,*

$$\dim_{\mathbb{C}}\left(\mathrm{Sol}(I)\right) \;=\; \mathrm{rank}(I)\,.$$

Example 2.6 (Example 2.28 revisited)**.** Let I be the D_2-ideal $\langle \partial_1 x_1 \partial_1,\, \partial_2^2{+}1\rangle$. Its holonomic rank is $\mathrm{rank}(I) = 4$. On simply connected domains outside $\mathrm{Sing}(I) = V(x_1)$, the solution space is 4-dimensional and equals

$$\mathrm{Sol}(I) \;=\; \mathbb{C}\cdot\big\{\sin(x_2),\, \cos(x_2),\, \log(x_1)\sin(x_2),\, \log(x_1)\cos(x_2)\big\}.$$

Indeed, the only singularities occurring are along the coordinate hyperplane $\{x_1 = 0\}$. ◇

Note that, in general, the singular locus is an set-theoretical upper bound for the actual singularities of the solution functions, as the following simple example demonstrates.

Example 2.7. The singular locus of $P = x\partial$ is the origin in the complex plane. However, all of its analytic solutions are entire functions. ◇

[5] Having the general theory such as the Riemann–Hilbert correspondence in mind, the solution functor has to be taken as the right-derived functor $\mathcal{S}\mathrm{ol}(\cdot) = \mathcal{RH}\mathrm{om}_{\mathcal{D}_X}(\cdot, \mathcal{O}_X^{\mathrm{an}})[\dim(X)]$, with a shift by the dimension of X. Their cohomology spaces are referred to as "higher solutions" of the $\mathcal{D}_X$-module. Note that these can be non-zero already for single $\mathcal{D}_X$-modules, i.e., complexes of $\mathcal{D}_X$-modules that are concentrated in degree 0.

2.2 Characteristic variety

To define the characteristic variety of a D-ideal, one needs to take initial forms of differential operators with respect to weight vectors. Weight vectors for the Weyl algebra D_n are allowed to be taken from the set

$$\mathcal{W} \;=\; \{(u,v)\in\mathbb{R}^{2n}\,|\, u_i+v_i\geq 0,\; i=1,\ldots,n\}. \tag{2.4}$$

The vector $(u,v)\in\mathcal{W}$ assigns weight u_i to x_i and weight v_i to ∂_i. Among others, $\mathcal{W}$ contains the set $\{(-w,w)|w\in\mathbb{R}^n\}$. Every operator P in the Weyl algebra has a unique expansion

$$P \;=\; \sum_{(\alpha,\beta)\,\in\, E} c_{\alpha,\beta}x^{\alpha}\partial^{\beta}, \tag{2.5}$$

called *normally ordered expression* of P, where $c_{\alpha,\beta}\in\mathbb{C}\backslash\{0\}$ and E is a finite subset of $\mathbb{N}^{2n}$. Unless stated otherwise, we always assume that differential operators are given in that form.

Definition 2.8. Fix a weight vector $(u,v)\in\mathcal{W}$. The (u,v)-*weight* of a differential operator $P\in D$ (expressed as in (2.5)) is the number $m=\max_{(\alpha,\beta)\in E}(\alpha\cdot u+\beta\cdot v)$. If (u,v) is of the form $(-w,w)$ for some $w\in\mathbb{R}^n$, the resulting number is called w-*weight* of P.

Example 2.9. Let $P=\theta_1+\theta_2+\theta_3+1\in D_3$ and $w=(-1,0,1)$. The w-weight of $\theta_1=x_1\partial_1$ is $(1,0,-1,-1,0,1)\cdot(1,0,0,1,0,0)=1-1=0$, and similarly for the remaining summands of P. The w-weight of P hence is $\max\{1-1,0+0,-1+1,0\}=0$. $\diamond$

Exercise 2.10. Consider the map $[x\partial,(\cdot)]\colon D_1\longrightarrow D_1,\; P\mapsto[x\partial,P]$, which sends a differential operator to its commutator with the Euler operator. Find a closed formula for this map which involves the w-weight for $w=1$. Is this map D-linear? $\diamond$

Each weight vector $(u,v)\in\mathcal{W}$ introduces an increasing filtration $F^{\bullet}_{(u,v)}(D)$ of the Weyl algebra, namely

$$\cdots\;\subseteq\; F^{k-1}_{(u,v)}(D)\;\subseteq\; F^{k}_{(u,v)}(D)\;\subseteq\; F^{k+1}_{(u,v)}(D)\;\subseteq\;\cdots$$

where $F^{k}_{(u,v)}(D)$ denotes the $\mathbb{C}$-vector space

$$F^{k}_{(u,v)}(D) \;=\; \left\{\sum_{\{(\alpha,\beta)\,|\,\alpha\cdot u+\beta\cdot v\leq k\}\text{ finite}} c_{\alpha,\beta}x^{\alpha}\partial^{\beta}\right\}. \tag{2.6}$$

They fulfill $F^{k}_{(u,v)}(D)\cdot F^{\ell}_{(u,v)}(D)\;\subseteq\;F^{k+\ell}_{(u,v)}(D)$, and the filtration is exhaustive, i.e., $\bigcup_k F^{k}_{(u,v)}(D)=D$. For $(u,v)=(1,\ldots,1)$, the resulting filtration is called *Bernstein filtration*. In this case, the $F^{k}_{(1,\ldots,1)}(D)$ are finite-dimensional. For $(u,v)=(0,1)$, where $0\in\mathbb{R}^n$ and 1 denotes the all-one vector in $\mathbb{R}^n$, the obtained filtration is the *order filtration* of D.

Remark 2.11. To obtain an increasing filtration as in (2.6), it is essential that $(u,v)\in\mathcal{W}$. Consider for instance $n=1$ and $(u,v)=(0,-1)\notin\mathcal{W}$. Then $\partial\in F^{-1}_{(0,-1)}(D)$, $x\in F^{0}_{(0,-1)}(D)$, but $\partial\cdot x=x\partial+1\notin F^{-1}_{(0,-1)}(D)$. $\diamond$

Exercise 2.12. Let $(u,v) \in \mathcal{W}$ and consider the induced filtration $F^{\bullet}_{(u,v)}(D)$. Prove that the associated graded ring $\mathrm{gr}_{(u,v)}(D) = \bigoplus_k F^k_{(u,v)}(D)/F^{k-1}_{(u,v)}(D)$ is

$$\mathrm{gr}_{(u,v)}(D) \;=\; \begin{cases} D & \text{if } u+v=0, \\ \mathbb{C}[x_1,\ldots,x_n,\xi_1,\ldots,\xi_n] & \text{if } u+v>0, \\ \text{a mixture of the above} & \text{otherwise.} \end{cases} \tag{2.7}$$

In (2.7), $u+v>0$ means $u_i+v_i>0$ for all $i=1,\ldots,n$. ◇

Definition 2.13. The *initial form* of a differential operator $P \in D \setminus \{0\}$ is

$$\mathrm{in}_{(u,v)}(P) \quad = \sum_{\alpha\cdot u+\beta\cdot v\,=\,m} c_{\alpha,\beta} \prod_{u_k+v_k>0} x_k^{\alpha_k}\xi_k^{\beta_k} \prod_{u_k+v_k\,=\,0} x_k^{\alpha_k}\partial_k^{\beta_k} \;\in\; \mathrm{gr}_{(u,v)}\,(D)\,, \tag{2.8}$$

and $\mathrm{in}_{(u,v)}(P)=0$ if $P=0$.

In the definition, the ξ_k are new variables that commute with all others. The initial form is an element in the graded ring $\mathrm{gr}_{(u,v)}(D)$. The case when u is the zero vector and v is the all-one vector $1=(1,\ldots,1)$ is of particular interest. Analysts refer to $\mathrm{in}_{(0,1)}(P)$ as the *principal symbol* of the differential operator P; it is an ordinary polynomial in $2n$ variables.

Example 2.14. The principal symbol of $P_1 = x^2\partial+1$ is $\mathrm{in}_{(0,1)}(P_1)=x^2\xi \in \mathbb{C}[x][\xi]$. The principal symbol of $P_2 = x_1\partial_1+x_2\partial_2+2x_2^7$ is $\mathrm{in}_{(0,1)}(P_2)=x_1\xi_1+x_2\xi_2 \in \mathbb{C}[x_1,x_2][\xi_1,\xi_2]$. ◇

A further important special case is if the weight vector is of the form $(u,v)=(-w,w)$ for some $w\in\mathbb{R}^n$. In that case, one denotes the initial form of P short-hand by $\mathrm{in}_w(P)$.

Exercise 2.15. Prove that for all $P,Q\in D_n$ and $w\in\mathbb{R}^n$, one has $\mathrm{in}_w(PQ)=\mathrm{in}_w(P)\cdot\mathrm{in}_w(Q)$ for the initial forms. ◇

Definition 2.16. Let $(u,v)\in\mathcal{W}$ and I a D-ideal. The *initial ideal* $\mathrm{in}_{(u,v)}(I)$ of I with respect to (u,v) is the $\mathrm{gr}_{(u,v)}(D)$-ideal generated by the initial forms of <u>all</u> elements of I. In symbols,

$$\mathrm{in}_{(u,v)}(I) \;=\; \big\langle\, \mathrm{in}_{(u,v)}(P) \,|\, P\in I \big\rangle \;\subset\; \mathrm{gr}_{(u,v)}(D)\,.$$

Example 2.17. For $I=\langle x_1\partial_2,\, x_2\partial_1\rangle \subset D_2$ and $(u,v)=(0,1)=(0,0,1,1)\in\mathbb{R}^4$, the initial ideal of I is the $\mathbb{C}[x_1,x_2][\xi_1,\xi_2]$-ideal

$$\mathrm{in}_{(0,1)}(I) \;=\; \big\langle\, x_1\xi_2,\, x_2\xi_1,\, x_1\xi_1-x_2\xi_2,\, x_2^2\xi_2,\, x_2\xi_2^2 \,\big\rangle\,.$$

This can be obtained by running the code below in `Singular:Plural` [13, 27]:[6]

```
LIB "dmod.lib";
def D2 = makeWeyl(2); setring D2; ideal I = x(1)*D(2), x(2)*D(1);
def CV = charVariety(I); setring CV; charVar;
```

[6] An online version of Singular is available at the following link: https://www.singular.uni-kl.de:8003/

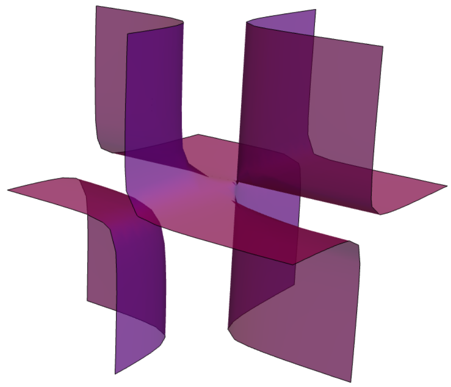


Figure 1: The set of real zeros of the graph polynomial of the massless parachute diagram with parameters $(p_1+p_2)^2 = 25$, $p_3^2 = 49$, $p_4^2 = 9$, as in [22]. Plotted with `Mathematica` [34].

The code uses the D-module libraries [2]. ⋄

Note bene. As the above example demonstrates, it is in general not sufficient to take only the initial forms of generators of the D-ideal into account. ⋄

Denote by $(0,1)$ the vector $(0,\dots,0,1,\dots,1) \in \mathbb{R}^{2n}$.

Definition 2.18. Let I be a D-ideal. The $\mathbb{C}[x_1,\dots,x_n,\xi_1,\dots,\xi_n]$-ideal $\mathrm{in}_{(0,1)}(I)$ is called the *characteristic ideal* of I.

We already saw an example of a characteristic ideal in Example 2.17.

An *affine algebraic variety* is the common vanishing set of a family of polynomials, endowed with the Zariski topology which is defined in terms of prime ideals. Sometimes, irreducibility is required. Note that ideals $\mathfrak{a} \subset \mathbb{K}[x_1,\dots,x_n]$ are not in one-to-one correspondence with their varieties, but the following is true for algebraically closed fields $\mathbb{K}$: $\sqrt{\mathfrak{a}} = I(V(\mathfrak{a}))$, where $\sqrt{\mathfrak{a}}$ denotes the radical ideal of $\mathfrak{a}$, and $I(V(\mathfrak{a}))$ is the ideal of $V(\mathfrak{a})$. This statement is the content of Hilbert's Nullstellensatz. To be able to take multiplicities into account, one would need to address the more refined theory of schemes of A. Grothendieck.

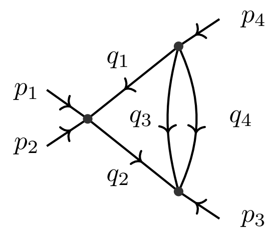
The algebraic variety of the characteristic ideal is one particularly important object associated to a D-ideal.

Definition 2.19. The *characteristic variety* of a D-ideal I is the vanishing set of the characteristic ideal, i.e.,

$$\mathrm{Char}(I) \coloneqq V\big(\mathrm{in}_{(0,1)}\big) = \big\{(x,\xi)\,|\,p(x,\xi)=0 \text{ for all } p \in \mathrm{in}_{(0,1)}(I)\big\} \subseteq \mathbb{C}^{2n}. \tag{2.9}$$

In order to remember the variables' names, one sometimes writes $\mathbb{C}^n_x \times \mathbb{C}^n_\xi$ for $\mathbb{C}^{2n}$ in (2.9).

Example 2.20 (Parachute diagram)**.** Consider the massless "parachute diagram" as in [22, Example 1], representing a 4-point scattering process:



Its Symanzik polynomials are

$$\mathcal{U}_G = \alpha_1\alpha_3 + \alpha_2\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_4 + \alpha_3\alpha_4\,,$$
$$\mathcal{F}_G = (p_1+p_2)^2\,\alpha_1\alpha_2\,(\alpha_3+\alpha_4) + p_3^2\alpha_2\alpha_3\alpha_4 + p_4^2\alpha_1\alpha_3\alpha_4 - \mathcal{U}_G \cdot \left(\sum_{i=1}^4 m_i^2\alpha_i\right).$$

The real points of $V(\mathcal{G}_G) \subset \mathbb{C}^4$ for $\alpha_4 = 1$ are plotted in Figure 1. $\diamond$

The irreducible components of an affine variety $V(I) \subset \mathbb{C}^n$ are obtained by the primary decomposition of its defining ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$. Recall that an ideal $\mathfrak{q} \subset \mathbb{C}[x_1, \ldots, x_n]$ is *primary* if, whenever $p \cdot q \in \mathfrak{q}$, it follows that $p \in \mathfrak{q}$ or $q \in \mathfrak{q}$ or $p, q \in \sqrt{\mathfrak{q}}$, where

$$\sqrt{\mathfrak{q}} = \{p \in \mathbb{C}[x_1, \ldots, x_n] \,|\, p^m \in \mathfrak{q} \text{ for some } m > 0\}$$

denotes the radical ideal of $\mathfrak{q}$. Every ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$ has an *irredundant* decomposition

$$I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_k \tag{2.10}$$

into primary ideals. The $\mathfrak{q}_i$ are not uniquely determined, but their underlying prime ideals $\mathfrak{p}_i := \sqrt{\mathfrak{q}_i}$ are; these are the *associated primes* of I. Irredundancy means that removing any of the primary ideals $\mathfrak{q}_i$ would change the intersection (2.10), and the $\sqrt{\mathfrak{q}_i}$ are pairwise distinct. The variety $V(I)$ has a unique irredundant decomposition into irreducible algebraic varieties

$$V(I) = \bigcup_{\mathfrak{p}_\mathrm{i} \text{ minimal}} V(\mathfrak{p}_\mathrm{i})\,.$$

Here, the union is taken over all associated prime ideals that are *minimal* over I.

Example 2.21. Consider the $\mathbb{C}[x, y, z]$-ideal

$$I = \left\langle (x+y+z-1)^2 \cdot (xz - (x+y)y) \right\rangle.$$

The variety $V(I)$ is plotted in Figure 2. The associated primary ideals of I and their underlying primes are $\langle (x+y+z-1)^2 \rangle$ with underlying prime $\langle x+y+z-1 \rangle$, and the prime ideal $(xz - (x+y)y)$. This can be obtained by running the following code in Singular.

```
LIB "primdec.lib";
ring r = 0,(x,y,z),dp; setring r;
ideal I = (x+y+z-1)^2*(x*z-(x+y)*y);
list pr = primdecGTZ(I); pr;
```

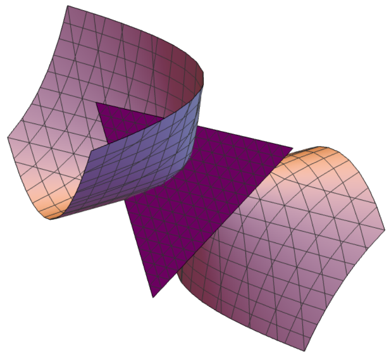


Figure 2: The real-valued points of the variety $V((x+y+z-1)^2(xz-(x+y)y))$.

The variety $V(I)$ decomposes into the purple surface and the mauve hyperplane. The curve obtained as their intersection has an interpretation as a discrete statistical model taking 3 states. This is the viewpoint of likelihood geometry [33]. $\diamond$

Exercise 2.22. Consider the D_1-ideal $I = \langle \partial^2, x\partial - 1\rangle$. Compute the characteristic ideal of I as well as its associated primes. $\diamond$

The following theorem was established by Sato, Kawai, and Kashiwara in [51].

Theorem 2.23 (Fundamental Theorem of Algebraic Analysis)**.** *Let $0 \subsetneq I \subsetneq D_n$ be a D-ideal. Every irreducible component of its characteristic variety* $\mathrm{Char}(I)$ *has dimension at least n.*

Definition 2.24. A D-ideal (or its associated D-module D/I) is called *holonomic* if the dimension of its characteristic ideal is n, i.e., as small as possible.

Exercise 2.25. Let $f = x_1^3 - x_2^2 \in \mathbb{C}[x_1, x_2]$ and consider the D_2-ideal

$$I \;=\; \Big\langle\, f\partial_1 + \frac{\partial f}{\partial x_1},\; f\partial_2 + \frac{\partial f}{\partial x_2}\,\Big\rangle\,.$$

Is I holonomic? Find a non-constant function that is annihilated by I. $\diamond$

Definition 2.26. A function $f(x_1,\ldots,x_n)$ is *holonomic* if its annihilator

$$\mathrm{Ann}_{D_n}(f) \;=\; \{P \in D_n \,|\, P \bullet f = 0\}$$

is a holonomic D_n-ideal.

Definition 2.27. The *holonomic rank* of a D_n-ideal I is

$$\mathrm{rank}(I) \;\coloneqq\; \dim_{\mathbb{C}(x)}\left(R_n/R_nI\right) \;=\; \dim_{\mathbb{C}(x)}\big(\mathbb{C}(x)[\xi]/\mathbb{C}(x)[\xi]\,\mathrm{in}_{(0,1)}(I)\big)\,. \tag{2.11}$$

The second equality in (2.11) follows from standard arguments in Gröbner basis theory.

Note bene. If I is holonomic, it follows that $\operatorname{rank}(I) < \infty$. The reverse implication is not true. To see this, revisit the non-holonomic D-ideal in Exercise 2.25. $\diamond$

Example 2.28. Consider the D_2-ideal $I = \langle \partial_1 x_1 \partial_1,\, \partial_2^2 + 1\rangle$. The holonomic rank of I is 4, since $\{1, \partial_1, \partial_2, \partial_1\partial_2\}$ is a basis of the $\mathbb{C}(x_1, x_2)$-vector space $R_2/R_2 I$. $\diamond$

Example 2.29 (GKZ systems)**.** Let A be an $n \times k$ integer matrix. Its *normalized volume*, denoted $\operatorname{vol}(A)$, is the volume of the union of the convex hull of the columns of A and the origin, scaled with respect to the standard n-simplex having volume 1. It is a lower bound for the holonomic rank of $H_A(\kappa)$: for all parameters $\kappa \in \mathbb{C}^n$, one has the inequality

$$\operatorname{rank}(H_A(\kappa)) \,\geq\, \operatorname{vol}(A)\,.$$

Equality holds for generic κ, but the identity may fail for special κ; see [50, Example 4.2.7]. $\diamond$

Exercise 2.30. Compute the holonomic rank of the GKZ system $H_A(\kappa)$ from Example 1.9 for a parameter $\kappa \in \mathbb{C}^3$ of your choice. Compare this number to $\operatorname{vol}(A)$. $\diamond$

Let I, J be ideals in a polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$. The *saturated ideal* $(I\colon J^\infty)$ with respect to J is the ideal

$$(I\colon J^\infty) \,\coloneqq\, \bigcup_{k\geq 1} \left(I\colon J^k\right),$$

where $(I\colon J^k)$ denotes the *ideal quotient* $\{p \in \mathbb{C}[x_1, \ldots, x_n] \,|\, pJ^k \subset I\}$. The variety of $(I\colon J^\infty)$ is the Zariski closure of $V(I) \setminus V(J)$. Geometrically, taking a saturation hence means to remove the component cut out by J (and takes the closure of the resulting set).

Exercise 2.31. Consider the $\mathbb{C}[x, y, z]$-ideals $I = \langle x^2yz\rangle$ and $J = \langle xy\rangle$. Compute $(I\colon J^\infty)$ and visualize their varieties. $\diamond$

Definition 2.32. The *singular locus* $\operatorname{Sing}(I)$ of I is the variety in $\mathbb{C}^n$ defined as

$$\operatorname{Sing}(I) \,\coloneqq\, V\left(\left(\operatorname{in}_{(0,1)}(I) : \langle \xi_1, \ldots, \xi_n\rangle^\infty\right) \cap \mathbb{C}[x_1, \ldots, x_n]\right).$$

Geometrically, the singular locus of I is the closure of the projection of $\operatorname{Char}(I)\backslash(\mathbb{C}^n \times \{0\})$ onto the first n coordinates of $\mathbb{C}^{2n} = \mathbb{C}^n_x \times \mathbb{C}^n_\xi$.

Remark 2.33. Outside the singular locus of a D-ideal I, the solutions to I form a vector bundle of rank $\operatorname{rank}(I)$. $\diamond$

If $I = DP$ for some $P \in D$, we sometimes also write $\operatorname{Sing}(P)$ for $\operatorname{Sing}(I)$. In case $n = 1$ and $I = DP$ for some $P = \sum_k a_k \partial^k \in D$, the singular locus is given be the vanishing locus of the polynomial $a_{\operatorname{ord}(P)}$. As a formula,

$$\operatorname{Sing}(P) \,=\, V(a_{\operatorname{ord}(P)}) \,=\, \{x \in \mathbb{C} \,|\, a_{\operatorname{ord}(P)}(x) = 0\}\,.$$

Example 2.34. Let $P_1 = x\partial^2 + \partial$ and $P_2 = x^2\partial + 1$. Their singular loci are equal, they are $\operatorname{Sing}(P_1) = \operatorname{Sing}(P_2) = \{0\} \subset \mathbb{C}$. $\diamond$

Example 2.35 (Example 2.28 revisited)**.** Let $I = \langle x_1\partial_1^2 + \partial_1, \partial_2^2 + 1\rangle$. A computation in Singular reveals that its characteristic ideal is $\langle \xi_2^2, x_1\xi_1^2\rangle \subset \mathbb{C}[x_1,x_2][\xi_1,\xi_2]$. Hence $\mathrm{Sing}(I)$ is the coordinate hyperplane $V(x_1) \subset \mathbb{C}^2$. ◇

Example 2.36. Let $I = \langle x_1\partial_2,\, x_2\partial_1\rangle \subset D_2$. Its singular locus is $\mathrm{Sing}(I) = V(x_1,x_2) = \{0\}$, and its solution space is $\mathrm{Sol}(I) = \mathbb{C}\cdot 1$. This simple example demonstrates that points in the singular locus may be singularities of its analytic solutions—but do not have to be. ◇

Example 2.37. The singular locus of a GKZ system $H_A(\kappa)$ is the variety cut out by the so called "principal A-determinant," see [24, Remark 1.8]. It is a product of individual discriminants to polynomial systems, one for each face of the cone over A; see [23, 50] for the precise statement, and the survey [48] of GKZ systems pointing to plenty of related work. In [20], inspired by GKZ discriminants, the authors contribute new algorithms to compute Landau singularities of Feynman integrals. ◇

3 Holonomic functions

Zeilberger [60] was the first to study holonomic functions in an algorithmic way. This lecture explains how to compute with holonomic functions and visits closure properties of this function class and is closely following the presentation in [37].

3.1 Weyl closure

We recall from Definition 2.26 that a function $f(x_1,\ldots,x_n)$ is *holonomic* if its annihilator

$$\mathrm{Ann}_{D_n}(f) \;=\; \{P \in D_n \,|\, P \bullet f = 0\}$$

is a holonomic D_n-ideal. We here are relaxed about the function class—typically, one imposes some analyticity assumption. In the univariate case, i.e., $n=1$, a function f is holonomic if and only if there exists a non-zero differential operator $P \in D \setminus \mathbb{C}[x]$ such that $P \bullet f = 0$.

Numerous functions in the sciences are holonomic, e.g., hypergeometric functions [48], Feynman integrals [12], many trigonometric functions, some probability distributions, many special functions such as Airy's or Bessel's functions, polylogarithms, or the volume of compact semi-algebraic sets [39]. By Theorem 2.5, holonomic function can be encoded by finite data, namely their annihilating D-ideal together with $\mathrm{rank}(\mathrm{Ann}_D(f))$-many initial conditions. This fact makes holonomic functions well-suited to be handed to computer, and to be investigated by means of computations with their annihilating D-ideal.

Exercise 3.1. Let $r \in \mathbb{C}(x)\setminus\{0\}$ be a non-zero rational function. Prove that the annihilator of r in the rational Weyl algebra R is generated by $r\partial - \frac{\partial r}{\partial x}$. ◇

Exercise 3.2. Determine a holonomic annihilating D_2-ideal of the function

$$f(x,y) \;=\; e^{x\cdot y}\cdot \sin\frac{y}{1+y^2}\,.$$

To do so, you may use the `Mathematica` package `HolonomicFunctions` [37]. The following code returns two annihilating differential operators of f.

```
<< RISC`HolonomicFunctions`
f = Exp[x*y]*Sin[y*1/(1+y^2)]
ann = Annihilator[f,{Der[x],Der[y]}]
```

It remains to investigate the D_2-ideal generated by them regarding holonomicity. $\diamond$

Sometimes, it is useful to slightly enlarge a considered D-ideal, namely by taking its Weyl closure, as was introduced by Tsai [56].

Definition 3.3. The *Weyl closure* of a D_n-ideal I is the D_n-ideal

$$W(I) \coloneqq R_n I \cap D_n \,. \tag{3.1}$$

A D-ideal is *Weyl-closed* if $W(I) = I$.

Clearly, for any D_n-ideal I, $R_n W(I) = R_n(I)$ as ideals in R_n.

Exercise 3.4. Let $I, J \subset D_n$ be D_n-ideals. Does $R_n I = R_n J$ as R_n-ideals imply that $I = J$ as D_n-ideals? Does $W(I) = W(J)$ imply that $I = J$? $\diamond$

Exercise 3.5. Let M be a D-module that is torsion-free as $\mathbb{C}[x_1, \ldots, x_n]$-module, and $f \in M$. Prove that $\mathrm{Ann}_D(f)$ is Weyl-closed. $\diamond$

Example 3.6. Consider $I = \langle x\partial \rangle \subset D_1$. Its Weyl closure is $I = \langle \partial \rangle$. The solution space of I is $\mathrm{Sol}(I) = \mathbb{C} \cdot 1$, i.e., the solutions are constant functions only. Although $\mathrm{Sing}(I) = \{0\}$, none of the solutions to I has a singularity there. Observe that $0 \notin \mathrm{Sing}(W(I))$. If we allow distributional solutions, we find that the Heaviside step function

$$H(x) \;=\; \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0, \end{cases}$$

is a solution to I, since its distributional derivative[7] is the Dirac delta δ, and $x \cdot \delta(x) \equiv 0$. $\diamond$

Example 3.7. Let $P = x\partial - n$, with $n \in \mathbb{N}$. Its solution space is $\mathrm{Sol}(P) = \mathbb{C} \cdot x^n$. The Weyl closure of $I = DP$ is $W(I) = \langle x\partial - n, \partial^{n+1} \rangle$. For this, note that $x\partial^{n+1} = \partial^n \cdot P$. In this particular case, taking the Weyl closure finds the full annihilator of $\mathrm{Sol}(P)$. $\diamond$

In general, it is a difficult task to compute the Weyl closure of a D-ideal.

Clearly, $I \subseteq W(I)$. Hence, for the singular locus and the space of holomorphic solutions to the system of PDEs encoded by I, one has

$$\mathrm{Sing}(I) \,\supseteq\, \mathrm{Sing}(W(I)) \quad \text{and} \quad \mathrm{Sol}(I) \,\supseteq\, \mathrm{Sol}(W(I))\,. \tag{3.2}$$

Moreover, $\mathrm{rank}(I) = \mathrm{rank}(W(I))$, since $R_n I = R_n W(I)$. Since every element Q of $W(I)$ can be written as $Q = r \cdot P$ for some $r \in \mathbb{C}(x_1, \ldots, x_n)$ and $P \in I$, we also have the inclusion

[7] Recall that for a Schwartz distribution $\psi \in (C_c^\infty(\mathbb{R}^n))'$ and a test function $f \in C_c^\infty(\mathbb{R}^n)$, one denotes $\langle \psi, f \rangle = \int_{\mathbb{R}^n} f\psi \,\mathrm{d}x$. Let $\alpha \in \mathbb{N}^n$. The *distributional derivative of ψ of order α* is the Schwartz distribution $\partial^\alpha \bullet \psi \in (C_c^\infty(\mathbb{R}^n))'$ defined by requiring $\langle \partial^\alpha \bullet \psi, f \rangle = (-1)^{|\alpha|} \langle \psi, \partial^\alpha \bullet f \rangle$ for any $f \in C_c^\infty(\mathbb{R}^n)$.

$\mathrm{Sol}(I) \subseteq \mathrm{Sol}(W(I))$. Hence, $\mathrm{Sol}(I) = \mathrm{Sol}(W(I))$. The first inclusion in (3.2) can be strict, which can already be seen for the D-ideal generated by $P = x\partial$:

$$\mathrm{Sing}(DP) \,=\, \{0\}, \quad \text{whereas} \;\; \mathrm{Sing}(W(I)) \,=\, \mathrm{Sing}(D\partial) \,=\, \emptyset\,.$$

In summary, a D-ideal I and its Weyl closure $W(I)$ have the same (classical) solution space, $W(I)$ might make the singular locus smaller, and might contain additional operators that annihilate all solutions of I. We summarize these insights in

Proposition 3.8. *Let I be a D_n-ideal and $W(I)$ its Weyl closure. Then*

(a) $\mathrm{Sol}(W(I)) = \mathrm{Sol}(I)\,,$ *(b)* $\mathrm{Sing}(W(I)) \subseteq \mathrm{Sing}(I)\,,$ *(c)* $W(I) \bullet \mathrm{Sol}(I) \,=\, 0\,.$

If a D-ideal I has finite holonomic rank, it follows from [50, Theorem 1.4.15] that its Weyl closure $W(I)$ is holonomic. To prove that a function $f(x_1,\ldots,x_n)$ is holonomic, it is therefore sufficient to find an annihilating D-ideal I of finite holonomic rank. If $I \subset \mathrm{Ann}_{D_n}(f)$ with $\mathrm{rank}(I) < \infty$, then its Weyl closure $W(I)$ is a holonomic D_n-ideal with $W(I) \subset \mathrm{Ann}_{D_n}(f)$. In particular, this forces $\mathrm{Ann}_{D_n}(f)$ to be holonomic.

Note bene. Even if $I \subset \mathrm{Ann}_D(f)$ is a holonomic D-ideal contained in the annihilator of a holonomic function f, the Weyl closure of I does not have to be equal to the full annihilator. Can you come up with an example that demonstrates this fact? ◇

Exercise 3.9. Compute the Weyl closure of the D_1-ideal generated by the operator

$$P \,=\, x^2(x-1)(x-3)\partial^2 \,-\, (6x^3 - 20x^2 + 12x)\partial \,+\, (12x^2 - 32x + 12)\,.$$

Compare the solution spaces and singular loci of I and $W(I)$. ◇

Proposition 3.10 ([25, Proposition 2.10])**.** *Let f be an element of a D-module M that is torsion-free as $\mathbb{C}[x_1,\ldots,x_n]$-module. Then the following three conditions are equivalent:*

(i) f is holonomic.

(ii) $\mathrm{rank}\,(\mathrm{Ann}_D(f)) < \infty$.

(iii) For each $i \in \{1,\ldots,n\}$ there exists an operator $P_i \in \mathbb{C}[x_1,\ldots,x_n]\langle\partial_i\rangle\backslash\{0\}$ that annihilates f.

Proof. Let $I = \mathrm{Ann}_D(f)$. If I is holonomic, then RI is a zero-dimensional ideal in R_n, i.e., $\dim_{\mathbb{C}(x)}(R_n/R_nI) < \infty$. This condition is equivalent to (ii) and (iii). For the implication from (ii) to (i), we note that $\mathrm{Ann}_D(f)$ is Weyl-closed, since M is torsion-free. Finally, $\mathrm{rank}\,(\mathrm{Ann}_D(f)) < \infty$ implies that $\mathrm{Ann}_D(f) = W(\mathrm{Ann}_D(f))$ is holonomic. □

Remark 3.11. Let $I = \mathrm{Ann}_D(f)$ be the annihilator of a holonomic function f, and fix a point $x_0 \in \mathbb{C}^n \setminus \mathrm{Sing}(I)$. Let $m_1,\ldots,m_n$ be the orders of the distinguished operators $P_1,\ldots,P_n \in I$ in Proposition 3.10 (iii). Thus, each P_k is a differential operator in ∂_k of order m_k whose coefficients are polynomials in $x_1,\ldots,x_n$. Suppose we impose initial conditions by specifying complex numbers for the $m_1m_2\cdots m_n$ many quantities

$$\big(\partial_1^{i_1}\cdots\partial_n^{i_n}\bullet f\big)|_{x=x_0} \quad \text{where } \; 0 \le i_k < m_k \; \text{ for } \; k=1,\ldots,n.$$

The operators $P_1,\ldots,P_n$ together with these initial conditions determine the function f uniquely within the vector space $\mathrm{Sol}(I)$. This specification is known as a *canonical holonomic representation* of f; see [60, Section 4.1]. ◇

3.2 Closure properties

From given holonomic functions, one can cook up many more. For instance, restrictions and definite integrals of holonomic functions are again holonomic; the construction of annihilating D-ideals is topic of Section 6. We here address some more closure properties of the class of holonomic functions.

Proposition 3.12. *If f, g are holonomic functions (on the same domain), then both their sum $f + g$ and their product $f \cdot g$ are holonomic functions as well.*

Proof. For each $i \in \{1, 2, \ldots, n\}$, there exist non-zero operators $P_i, Q_i \in \mathbb{C}[x]\langle\partial_i\rangle$, such that $P_i \bullet f = Q_i \bullet g = 0$. Set $n_i = \operatorname{ord}(P_i)$ and $m_i = \operatorname{ord}(Q_i)$. The $\mathbb{C}(x)$-span of $\left\{\partial_i^k \bullet f\right\}_{k=0,\ldots,n_i}$ is a vector space of dimension $\leq n_i$. Similarly, the $\mathbb{C}(x)$-span of the set $\left\{\partial_i^k \bullet g\right\}_{k=0,\ldots,m_i}$ has dimension $\leq m_i$. Now consider $\partial_i^k \bullet (f + g) = \partial_i^k \bullet f + \partial_i^k \bullet g$. The $\mathbb{C}(x)$-span of $\left\{\partial_i^k \bullet (f + g)\right\}_{k=0,\ldots,n_i+m_i}$ has dimension $\leq n_i + m_i$. Hence, there exists a non-zero operator $S_i \in \mathbb{C}[x]\langle\partial_i\rangle$, such that $S_i \bullet (f + g) = 0$. Since this holds for all indices i, we conclude that the sum $f + g$ is holonomic. A similar proof works for the product $f \cdot g$. For each $i \in \{1, 2, \ldots, n\}$, we now consider the set $\left\{\partial_i^k \bullet (f \cdot g)\right\}_{k=0,1,\ldots,n_i m_i}$. By applying Leibniz's rule for taking derivatives of a product, we find that the $m_i n_i + 1$ elements of this set are linearly dependent over the field $\mathbb{C}(x)$. Hence, there is a non-zero operator $T_i \in \mathbb{C}[x]\langle\partial_i\rangle$ such that $T_i \bullet (f \cdot g) = 0$. We conclude that $f \cdot g$ is holonomic. $\square$

Proposition 3.13. *Let $f(x_1, \ldots, x_n)$ be holonomic. Then its partial derivatives $\partial_i \bullet f$, $i = 1, \ldots, n$, are holonomic functions as well.*

Proof. Since f is holonomic, there exist $P_i \in \mathbb{C}[x_1, \ldots, x_n]\langle\partial_i\rangle$, $i = 1, \ldots, n$, such that $P_i \bullet f = 0$. Rewrite each $P_i = \tilde{P}_i \partial_i + a_i$ with $a_i \in \mathbb{C}[x_1, \ldots, x_n]$. If $a_i = 0$, it follows $\tilde{P}_i \bullet (\partial_i \bullet f) = 0$. Now assume that a_i is not the zero polynomial. Since both a_i and f are holonomic, so is their product and there exists $Q_i \in \mathbb{C}[x_1, \ldots, x_n]\langle\partial_i\rangle$ which annihilates their product. Then $Q_i \tilde{P}_i \bullet (\partial_i \bullet f) = Q_i \bullet (-a_i f) = 0$, proving that $\partial_i \bullet f$ is holonomic. $\square$

Proposition 3.14. *Let $f(x)$ be a holonomic function. Its reciprocal $1/f$ is holonomic if and only if its logarithmic derivative, f'/f, is algebraic.*

For a proof, we refer to work of Harris and Sibuya [29]. This implies that, for instance, the function $1/\sin$ is not holonomic. At the same time, this shows that, in general, the composition of two holonomic functions is in general not holonomic. But there is a partial rescue.

Proposition 3.15. *Let $f(x)$ be holonomic and $g(x)$ algebraic. Then their composition $f(g(x))$ is a holonomic function.*

Proof. Let $h \coloneqq f \circ g$. By the chain rule, all derivatives $h^{(i)}$ can be expressed as linear combinations of $f(g), f'(g), f''(g), \ldots$ with coefficients in $\mathbb{C}[g, g', g'', \ldots]$. Since g is algebraic, it fulfills some polynomial equation $G(g, x) = 0$. By taking derivatives of this equation, we can express each $g^{(i)}$ as a rational function of x and g. We conclude that the ring $\mathbb{C}[g, g', \ldots]$ is contained in the field $\mathbb{C}(x, g)$. Denote by W the vector space spanned by $f(g), f'(g), \ldots$ over

$\mathbb{C}(x, g)$ and by V the vector space spanned by $f, f', \dots$ over $\mathbb{C}(x)$. Since f is holonomic, V is finite-dimensional over $\mathbb{C}(x)$. This implies that W is finite-dimensional over $\mathbb{C}(x, g)$. Since g is algebraic, $\mathbb{C}(x, g)$ is finite-dimensional over $\mathbb{C}(x)$. It follows that W is a finite-dimensional vector space over $\mathbb{C}(x)$, hence $h = f \circ g$ is holonomic. $\square$

4 Gröbner bases in Weyl algebras

We now learn about Gröbner bases in Weyl algebras, certain sets of generators of D-ideals that depend on the choice of a term order on the set of monomials. They enable practitioners of algebraic analysis to manipulate D-ideals in computer algebra software.

4.1 Term orders in the Weyl algebra

Let $(u, v) \in \mathcal{W}$ be a weight vector for the Weyl algebra with $\mathcal{W}$ as defined in (2.4). For a subset $G \subset D$ of the Weyl algebra, we denote by

$$\mathrm{in}_{(u,v)}(G) \;=\; \{\mathrm{in}_{(u,v)}(P) \,|\, P \in G\} \;\subset\; \mathrm{gr}_{(u,v)}(D)$$

the set containing the initial forms of all elements of G.

Definition 4.1. Let I be a D-ideal. A finite set $G \subset D$ of differential operators is a *Gröbner basis of I with respect to* (u, v) if I is generated by G and if the initial ideal of I with respect to (u, v) is generated by the initial forms of elements in G, i.e., if

$$\mathrm{in}_{(u,v)}(I) \;=\; \big\langle\, \mathrm{in}_{(u,v)}(G) \big\rangle$$

is an equality of $\mathrm{gr}_{(u,v)}(D)$-ideals.

Example 4.2. The characteristic ideal in Example 2.17 provides an example of a generating set which is not a Gröbner basis with respect to the weight vector $(0, 1) \in \mathbb{R}^4$. $\diamond$

We will again use the normally ordered expression of differential operators from (2.5), i.e., we write $P \in D$ in the form

$$P \;=\; \sum_{(\alpha,\beta)\,\in\, E} c_{\alpha,\beta} x^{\alpha} \partial^{\beta}. \tag{4.1}$$

To speak about Gröbner bases, we will need to define total a total order $\prec$ on the set of monomials $x^{\alpha}\partial^{\beta}$ in D. Such an order is called a *multiplicative monomial order* if both

(1) $1 \prec x_i\partial_i$ for $i = 1, \dots, n$ and

(2) $x^{\alpha}\partial^{\beta} \prec x^{a}\partial^{b}$ implies $x^{\alpha+s}\partial^{\beta+t} \prec x^{a+s}\partial^{b+t}$ for all $(s, t) \in \mathbb{N}^n \times \mathbb{N}^n$.

Now let a multiplicative monomial order $\prec$ be fixed.

Definition 4.3. The *initial monomial* $\mathrm{in}_\prec(P)$ of a differential operator $P \in D$ is the monomial $x^\alpha \xi^\beta \in \mathbb{C}[x,\xi]$ such that $x^\alpha \partial^\beta$ is the $\prec$-largest monomial occurring in (4.1). The *initial ideal* $\mathrm{in}_\prec(I)$ of a D-ideal I is the monomial $\mathbb{C}[x,\xi]$-ideal generated by $\{\mathrm{in}_\prec(P) \,|\, P \in I\}$.

Definition 4.4. A finite set $G \subset D$ is a *Gröbner basis of* I *with respect to* $\prec$ if I is generated by G and $\mathrm{in}_\prec(I)$ is generated by $\mathrm{in}_\prec(G) = \{\mathrm{in}_\prec(P) \,|\, P \in G\}$.

Note bene. Gröbner bases are in general not unique—they are if one imposes that the Gröbner basis is "reduced," see [50, Definition 1.1.12]. $\diamond$

A multiplicative monomial order $\prec$ is called a *term order* for D if $1 = x^0\partial^0$ is the smallest element. This condition arises from the commutator relation (1.1) and guarantees compatibility with multiplication in the sense that $\mathrm{in}_\prec(PQ) = \mathrm{in}_\prec(P)\cdot\mathrm{in}_\prec(Q)$. For term orders, there are no infinitely decreasing chains in D. Examples of term orders are the lexicographic order, the reverse lexicographic order, elimination orders, and the graded reverse lexicographic order, typically built upon $\partial_1 \succ \partial_2 \succ \cdots \partial_n \succ x_1 \succ \cdots \succ x_n$.

We here recall the lexicographic and reverse lexicographic order for (commutative!) polynomial rings from [9, §2], which is one of the standard references for Gröbner bases.

Example 4.5 (`lex`)**.** Let $S = \mathbb{C}[x_1,\ldots,x_n]$ be a polynomial ring and $\alpha, \beta \in \mathbb{N}^n$. The *lexicographic order* is the following order:

$$x^\alpha \succ x^\beta \text{ if the } \underline{\text{leftmost}} \text{ non-zero entry of } \alpha - \beta \in \mathbb{Z}^n \text{ is } \underline{\text{positive}}.$$

For instance, $x_1 \succ x_2 \succ \cdots \succ x_n$, $x_1x_2^2 \succ x_2^3x_3^4$, and $x_1^3x_2^2x_3^4 \succ x_1^3x_2^2x_3$. $\diamond$

Example 4.6 (`revlex`)**.** Let $S = \mathbb{C}[x_1,\ldots,x_n]$ be a polynomial ring and $\alpha, \beta \in \mathbb{N}^n$. The *reverse lexicographic order* is the following order:

$$x^\alpha \succ x^\beta \text{ if the } \underline{\text{rightmost}} \text{ non-zero entry of } \alpha - \beta \in \mathbb{Z}^n \text{ is } \underline{\text{negative}}.$$

For instance, again $x_1 \succ x_2 \succ \cdots \succ x_n$ and $x_1x_2^2 \succ x_2^3x_3^4$, but $x_1^3x_2^2x_3^4 \prec x_1^3x_2^2x_3$. $\diamond$

We now come back to monomials in the Weyl algebra.

Example 4.7 (`degrevlex`)**.** Let $\alpha, \beta, a, b \in \mathbb{N}^n$. We denote $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and likewise for β, a, b. The *degree reverse lexicographic order* is the order defined as follows:

$$x^\alpha\partial^\beta \succ x^a\partial^b \text{ if } |\alpha| + |\beta| > |a| + |b|, \text{ or } |\alpha| + |\beta| = |a| + |b| \text{ and the rightmost}$$
$$\text{non-zero entry of } (\alpha,\beta) - (a,b) \in \mathbb{N}^{2n} \text{ is negative.}$$

For instance, $x_1^2x_2\partial_2^2 \succ x_1x_2\partial_2^2 \succ x_2\partial_1\partial_2^2$. $\diamond$

These definitions naturally extend to $\mathrm{gr}_{(u,v)}(D)$-ideals.

We now have two different notions of Gröbner bases of D-ideals: one with respect to weight vectors, and another one with respect to multiplicative monomial orders. They are related as follows. Let $(u,v) \in \mathcal{W}$ and let $\prec$ be any term order. The order $\prec_{(u,v)}$ is the multiplicative monomial defined as follows:

$$x^\alpha\partial^\beta \prec_{(u,v)} x^a\partial^b \Leftrightarrow \alpha u + \beta v < au + bv \text{ or } (\alpha u + \beta v = au + bv \text{ and } x^\alpha\partial^\beta \prec x^a\partial^b),$$

i.e., we use $\prec$ as a tiebreaker in case two monomials have the same (u,v)-weight. This defines a term order if and only if (u,v) is non-negative.

Theorem 4.8 ([50, Theorem 1.1.6]). *Let I be a D-ideal, $(u,v) \in \mathcal{W}$ any weight vector, $\prec$ any term order, and G a Gröbner basis of I with respect to $\prec_{(u,v)}$. Then*

(1) the set G is a Gröbner basis of I with respect to (u,v) and

(2) the set $\mathrm{in}_{(u,v)}(G)$ is a Gröbner basis of $\mathrm{in}_{(u,v)}(I)$ with respect to $\prec$.

Theorem 4.9 ([50, Theorem 1.1.7]). *Let $\prec$ be a term order on D and $G = \{G_1, \ldots, G_k\}$ a Gröbner basis for its D-ideal $I = D \cdot G$ with respect to $\prec$. Any $P \in I$ admits a standard representation in terms of G: there exist $C_1, \ldots, C_m \in D$ such that*

$$P = \sum_{i=1}^{k} C_i G_i, \quad \textit{where } G_i \in G \textit{ and } \mathrm{in}_\prec(C_i G_i) \preceq \mathrm{in}_\prec(P) \textit{ for all } i.$$

The C_i's are in general not unique. The proof of Theorem 4.9 uses the *normal form algorithm* (also called *division algorithm*), presented in [50, p. 7]. Fix a multiplicative monomial order $\prec$ on D, a subset $G = \{G_1, \ldots, G_k\}$ of the Weyl algebra, and let $P \in D$.

Algorithm 4.10 (Normal form algorithm).
$\mathrm{normalForm}_\prec(P, \{G_1, \ldots, G_k\}) \coloneqq$
  $R \coloneqq P$
  while $(\mathrm{in}_\prec(R)$ is divisible by an $\mathrm{in}_\prec(G_i))$ {
    $R \coloneqq \mathrm{sp}(\mathrm{R}, \mathrm{G_i})$
  }
  $R \coloneqq \mathrm{in}_\prec(R)_{\xi \to \partial} + \mathrm{normalForm}_\prec(R - \mathrm{in}_\prec(R)|_{\xi \to \partial}, \{G_1, \ldots, G_k\})$
  return(r)

When $\prec$ is a term order, Algorithm 4.10 terminates. In general, the normal form is not unique, since different elements of G might be used for the reduction steps. Gröbner bases provide a rescue: if G is a Gröbner basis with respect to a $\prec$ term order, the normal form is unique. In particular, for any Gröbner basis G of I, the normal form of $P \in I$ by G is 0.

4.2 Computing Gröbner bases

A good portion of "classical" Gröbner basis theory for commutative polynomial rings can be carried over to the non-commutative setup. However, some adaptions need to be made and there are subtle differences. For instance, in the case of polynomial rings, *Buchberger's Criterion 1* states that a set of polynomials whose initial monomials have disjoint support is automatically a Gröbner basis. This criterion does not hold in the Weyl algebra. For example, consider $P = \underline{\partial_2} + x_1, Q = \underline{\partial_1}$ and a term order $\prec$ which selects the underlined monomials as initial terms. Here, Buchberger's Criterion 1 would imply that $\{P, Q\}$ is a Gröbner basis, but this is clearly false since $QP - PQ = 1$ and hence $\mathrm{in}_\prec(\langle P, Q\rangle) = \mathbb{C}[x_1, x_2, \xi_1, \xi_2] \supsetneq \langle \xi_1, \xi_2 \rangle$, which is a contradiction to $\{P, Q\}$ being a Gröbner basis.

A criterion that is valid in the Weyl algebra is based on "S-pairs," whose definition we recall now. Let $\prec$ be a term order on D. Let

$$\begin{aligned} D \ni P &= P_{\alpha\beta} x^\alpha \partial^\beta + \text{lower order terms} \ \text{ w.r.t. } \prec, \\ D \ni Q &= Q_{ab} x^a \partial^b + \text{lower order terms} \ \text{ w.r.t. } \prec \end{aligned}$$

be in normally ordered form. The *S-pair* of P and Q is

$$\mathrm{sp}(P,Q) \coloneqq x^{\alpha'}\beta^{\partial'}P - \frac{P_{\alpha\beta}}{Q_{ab}}x^{a'}\partial^{b'}Q, \tag{4.2}$$

where $\alpha_i' = \max(\alpha_i, a_i) - \alpha_i$, $\beta_i' = \max(\beta_i, b_i) - \beta_i$, $a_i' = \max(\alpha_i, a_i) - a_i$, $b_i' = \max(\beta_i, b_i) - b_i$. The multipliers in the S-pair are chosen s.t. the initial monomials of P and Q cancel out.

The following algorithm to determine a Gröbner basis is Algorithm 1.1.9 in [50].

Algorithm 4.11 (Buchberger algorithm)**.**
Input: $P = \{P_1, \ldots, P_k\}$ a subset of D, $\prec$ a term order.
Output: G, a Gröbner basis for DP w.r.t. $\prec$.
pair $\coloneqq \{(P_i, P_j) \,|\, 1 \leq i < j \leq k\}$
$G \coloneqq P$
while (pair $\neq \emptyset$) {
  Take any element (P, P') from the set pair.
  pair $\coloneqq$ pair $\setminus \{(P, P')\}$
  $h \coloneqq \mathrm{sp}(P, P')$
  $r \coloneqq \mathrm{normalForm}_{\prec}(h, G)$
  if ($r \neq 0$) {
    pair $\coloneqq$ pair $\cup \{(g, r) | g \in G\}$
    $G \coloneqq G \cup \{r\}$ }
}
return(G)

Theorem 4.12 ([50, Theorem 1.1.10])**.** *Let $G = \{G_1, \ldots, G_k\}$ be a finite subset of D and $\prec$ a term order on D.*

1. *(S-pair criterion) The set G is a Gröbner basis of $I = DG$ with respect to $\prec$ if and only if for all pairs $i \neq j$, the normal form of the S-pair* $\mathrm{sp}(\mathrm{G_i}, \mathrm{G_j})$ *by G is zero.*
2. *The Buchberger algorithm terminates and outputs a Gröbner basis of I w.r.t. $\prec$.*

5 Writing systems of linear PDEs in matrix form

This section explains how to write systems of (higher-order) linear PDEs as a first-order matrix system in a systematic way, namely with the help of Gröbner bases.s

5.1 Gröbner bases in the rational Weyl algebra

The computations in this section are carried out in the rational Weyl algebra R_n. We will denote $\mathbb{C}(x) = \mathbb{C}(x_1, \ldots, x_n)$ for brevity. We first need to relate Gröbner bases in D_n and R_n.

Definition 5.1. A term order $\prec$ on D_n is an *elimination term order* if $\partial^\beta \prec \partial^\gamma$ implies $x^\alpha \partial^\beta \prec \partial^\gamma$ for all $\alpha \in \mathbb{N}^n$.

Example 5.2. The $(0, v)$-weight with positive $v \in \mathbb{R}^n_{>0}$ refined by `lex` based on $\partial_1 \succ \partial_2 \succ \cdots \succ \partial_n \succ x_1 \succ \cdots \succ x_n$ is an elimination term order. $\diamond$

Exercise 5.3. Is the reverse lexicographic order an elimination term order on D_n? $\diamond$

Let $\prec$ be a term order on D_n. We will denote by $\prec'$ its restriction to monomials in the ∂_i's; this is a term order on $\mathbb{N}^n$. For any choice of elimination term order $\prec$ on D_n, also $\prec_{(0,1)}$ and $\prec_{(0,v)}$ with $v \in \mathbb{R}^n_{>0}$ are is an elimination term order on D_n. If G is a Gröbner basis of a D_n-ideal I with respect to an elimination order $\prec$ on D_n, then G is also a Gröbner basis of the the R_n-ideal $R_n I$ with respect to the order $\prec'$. A generating set of the initial ideal of $R_n I$ is obtained by replacing each of the variables $x_1, \ldots, x_n$ in the initial ideal of I by 1. By standard arguments in Gröbner basis theory, the holonomic rank (2.11) of a D_n-ideal I hence is the number of standard monomials of I considered as a $\mathbb{C}(x)[\xi]$-ideal.

5.2 Pfaffian systems

We are now ready to turn to Pfaffian systems. Let I a D_n-ideal with finite holonomic rank. Thus, $R_n/R_n I$ is finite-dimensional over $\mathbb{C}(x_1, \ldots, x_n)$. Let $\operatorname{rank}(I) = m \in \mathbb{N}_{>0}$, and write $S = \{s_1, \ldots, s_m\}$ for the set of standard monomials for a Gröbner basis of $R_n I$ in R_n. We can assume that $s_1 = 1$. For $f \in \operatorname{Sol}(I)$, denote by F the vector

$$F \coloneqq (s_1 \bullet f, s_2 \bullet f, \ldots, s_m \bullet f)^\top$$

of holonomic functions. Since the D_n-ideal I has holonomic rank m, there exist unique matrices $P_1, \ldots, P_n \in \mathbb{C}(x_1, \ldots, x_n)^{m \times m}$ such that

$$\partial_i \bullet F = P_i \cdot F \qquad \text{for} \quad i = 1, \ldots, n \tag{5.1}$$

for <u>any</u> $f \in \operatorname{Sol}(I)$. The system of first-order linear PDEs in (5.1) the *Pfaffian system of f*. In slight abuse of notation,[8] we will refer to the matrices P_i's as "connection matrices."

Exercise 5.4. Prove that the connection matrices fulfill the *integrability conditions*, i.e., $P_i P_j - P_j P_i = \partial_i \bullet P_j - \partial_j \bullet P_i$ for all i, j, where entry-wise differentiation of the connection matrices is meant. $\diamond$

Note bene. Left multiplication by ∂_i is only a $\mathbb{C}$-linear endomorphism on $R_n/R_n I$, it is not $\mathbb{C}(x)$-linear. The connection matrices hence do not encode ∂_i in the naïve linear algebra sense. Instead, they need to be extended using the Leibniz rule. In particular, taking mixed partial derivatives is not encoded by the respective powers of the connection matrices, since these may contain rational functions in the x-variables. $\diamond$

[8] Generically, holonomic D-modules are vector bundles endowed with a flat (also called "integrable") connection, which encodes the action of Weyl algebra. The matrices P_i in the Pfaffian system of I are in fact the connection matrices of the dual of the D-module D/I, when expressing the connection matrices in the respective basis. For the sake of these lecture notes, this technical detail is swept under the rug.

A change of basis $\widetilde{F} = GF$, with $G \in \mathbb{C}(x)^{m\times m}$ invertible, results in a *gauge transformation* of the connection matrices. The transformed system is $\partial_i \bullet \widetilde{F} = \widetilde{P}_i \cdot \widetilde{F}$ for

$$\widetilde{P}_i \;=\; \frac{\partial G}{\partial x_i} G^{-1} + G P_i G^{-1}\,. \tag{5.2}$$

Regular singular systems can be brought into Fuchsian form via a suitable gauge transform, i.e., the resulting matrices having poles of order at most one.

Example 5.5 ($n = 1$)**.** Let f be a holonomic function annihilated by the D_1-ideal

$$I \;=\; \left\langle x\partial^3 - (x+1)\partial + 1 \right\rangle .$$

The generator by itself is a Gröbner basis for $R_1 I$. The set of standard monomials equals $S = \{1, \partial, \partial^2\}$, and this is a $\mathbb{C}(x)$-basis of $R_1/R_1 I$. From I we see that

$$\partial^3 \bullet f \;=\; \frac{x+1}{x}\partial \bullet f - \frac{1}{x} \cdot f\,.$$

Let $F = (f, \partial \bullet f, \partial^2 \bullet f)^T$. This yields the following Pfaffian system for f:

$$\partial \bullet F \;=\; P \cdot F\,, \quad \text{where} \quad P \;=\; \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{x} & \frac{x+1}{x} & 0 \end{bmatrix}$$

is the *companion matrix*. For any non-zero real number u, we have

$$\begin{bmatrix} f'(u) \\ f''(u) \\ f'''(u) \end{bmatrix} \;=\; \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{u} & \frac{u+1}{u} & 0 \end{bmatrix} \cdot \begin{bmatrix} f(u) \\ f'(u) \\ f''(u) \end{bmatrix} .$$

This matrix-vector formula is useful for the design of numerical algorithms such as the holonomic gradient method and descent [53]. $\diamond$

Connection matrices hence are a multivariate analog of companion matrices of ODEs. Pfaffian systems depend on the chosen term order on R_n and on the R_n-ideal $R_n I$ only. In particular, the D-ideal I may be replaced by its Weyl closure $W(I)$ as defined in (3.1).

The connection matrices can be computed as follows. We apply the division algorithm modulo our Gröbner basis of $R_n I$ to the operators $\partial_i s_j$, where $i = 1, \ldots, n$ and $j = 1, \ldots, m$. The resulting normal form equals

$$a_{j1}^{(i)}(x) s_1 \;+\; a_{j2}^{(i)}(x) s_2 \;+\; \cdots \;+\; a_{jm}^{(i)}(x) s_m\,,$$

where the coefficients $a_{jk}^{(i)}$ are rational functions in $x_1, \ldots, x_n$. This means that the operator

$$\partial_i s_j - \sum_{k=1}^{m} a_{jk}^{(i)}(x) s_k$$

is contained in the R_n-ideal R_nI. From this, one sees that $a_{jk}^{(i)}(x)$ is the (j,k)-th entry of the connection matrix P_i.

The computation of Pfaffian systems for D-ideals is implemented in the package `ConnectionMatrices` [28], which is contained in the latest release of the open source computer algebra software *Macaulay2* [26]. The rational Weyl algebra is not implemented in *Macaulay2*. The main step was hence the implementation of a reduction algorithm in R_n, for which we took a detour via Gröbner bases in the Weyl algebra D_n. The documentation of the package also contains the example of differential equations for a massless triangle diagram as studied in [30] and for correlator function in theoretical cosmology as studied in [21] together with a gauge transformation to ε-factorized form.

A different method for the computation of Pfaffian systems can be found in [8], in which the authors provide efficient algorithms based on "Macaulay matrices."

Example 5.6 $(n=2)$**.** Let $I = \langle x\partial_x^2 - y\partial_y^2 + \partial_x - \partial_y, x\partial_x + y\partial_y + 1\rangle \subset D_2$. Let $\prec$ be the reverse lexicographic order on D_2.

(i) Let $w_1 = (0,0,2,1)$ and consider the elimination order $\prec_{w_1}$ on D_n. The standard monomials of R_2I w.r.t. $\prec'_{w_1}$ are $s_0 = 1, s_1 = \partial_y$, which yields the connection matrices

$$P_1 = \begin{bmatrix} -\frac{1}{x} & -\frac{y}{x} \\ -\frac{1}{x(x-y)} & -\frac{x+y}{x(x-y)} \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} 0 & 1 \\ \frac{1}{(x-y)y} & \frac{3y-x}{(x-y)y} \end{bmatrix}.$$

For $(0,0,1,1)$, one obtains the same standard monomials and connection matrices.

(ii) Let $w_2 = (0,0,1,2)$. The standard monomials of R_2I w.r.t. $\prec'_{w_2}$ are $s_0 = 1, s_1 = \partial_x$, and the connection matrices are

$$P_1 = \begin{bmatrix} 0 & 1 \\ -\frac{1}{(x-y)x} & \frac{y-3x}{x(x-y)} \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} -\frac{1}{y} & -\frac{x}{y} \\ \frac{1}{(x-y)y} & \frac{x+y}{(x-y)y} \end{bmatrix}.$$

$\diamond$

Exercise 5.7. Let I be as in Example 5.6.

(a) Compute $\mathrm{Sol}(I)$ and $\mathrm{Sing}(I)$.

(b) Reproduce the Pfaffian systems for the different choices of weight vectors using the package `ConnectionMatrices.m2` [28].

(c) Determine the gauge matrix that translates between the connection matrices for the different weight vectors, refined by the reverse lexicographic order, in (i) and (ii).

You are also encouraged to compute Pfaffian systems of differential equations behind an involved Feynman integral of your choice. We'd be excited to learn about your findings. $\diamond$

In the business of dimensional regularization, the resulting system of differential equations depends on an additional parameter ε defined by $D = 4 - 2\varepsilon$, for D the dimension of the considered Minkowski spacetime. The respective Weyl algebra then is $D_n(\varepsilon) = \mathbb{C}(\varepsilon)[x_1, \ldots, x_n]\langle\partial_1, \ldots, \partial_n\rangle$. Connection matrices that are in *ε-factorized form*, i.e. of the form $A = \varepsilon\widetilde{A}$ with $\widetilde{A}$ independent of ε, are particularly popular in practice. Series solutions to such systems in the small parameter ε can be computed as iterated integrals via the path-ordered exponential formalism. The fact of being ε-factorized also has strong implications on the connection matrices: in this case, they are guaranteed to commute. If the entries of the connection matrix are moreover logarithmic derivatives of rational function, they are referred to as being in "canonical form" [31]. Some methods for the construction of such a form are provided in [15] and the references pointed out therein. Also higher genus is starting being addressed [17]. We here gave a tiny sample of related work only; this exposition is by far not exhaustive and undoubtedly not giving credit to everyone who contributed.

6 Operations on D-modules

This section addresses how certain integral transforms, namely the Fourier–Laplace and the Mellin transform, affect the annihilating D-ideals of the original functions. Section 6.2 explains the operations of integrating and restricting to subspaces, capturing special cases of D-module theoretic direct and inverse images.

6.1 Integral transform

The Fourier(–Laplace) transform of a function $f\colon \mathbb{R}_{>0} \to \mathbb{C}$ is

$$\mathcal{F}\{f\}(t) \;=\; \int_{\mathbb{R}_{>0}} f(x)e^{-xt}\,\mathrm{d}x\,.$$

This integral converges if $f \in L^1$. Assuming suitable vanishing conditions on the boundary of the integration domain,[9] one reads that

$$\mathcal{F}\{x\cdot f\}(t) \;=\; -\partial_t \bullet \mathcal{F}\{f\}(t) \quad\text{and}\quad \mathcal{F}\{\frac{\partial f}{\partial x}\}(t) \overset{\text{IBP}}{=} t\cdot\mathcal{F}\{f\}(t)\,, \tag{6.1}$$

where the second identity follows from integration by parts (IBP).

Algebraically, the Fourier–Laplace transform is the isomorphism of Weyl algebras

$$\begin{gathered}\mathcal{F}\{\cdot\}\colon \mathbb{C}[x_1,\ldots,x_n]\langle\partial_1,\ldots,\partial_n\rangle \longrightarrow \mathbb{C}[t_1,\ldots,t_n]\langle\partial_{t_1},\ldots,\partial_{t_n}\rangle,\\ x_i\mapsto -\partial_{t_i},\quad \partial_i\mapsto t_i\,,\end{gathered} \tag{6.2}$$

reflecting the rules in (6.1).

[9] One possible option is to assume that f has rapid decay at 0 and ∞, i.e., it decays faster than any power of x, as x approaches 0 or ∞. By adapting the integration contour, one can get rid of the strict assumption of rapid decay. Borel–Moore homology constitutes the right framework for that.

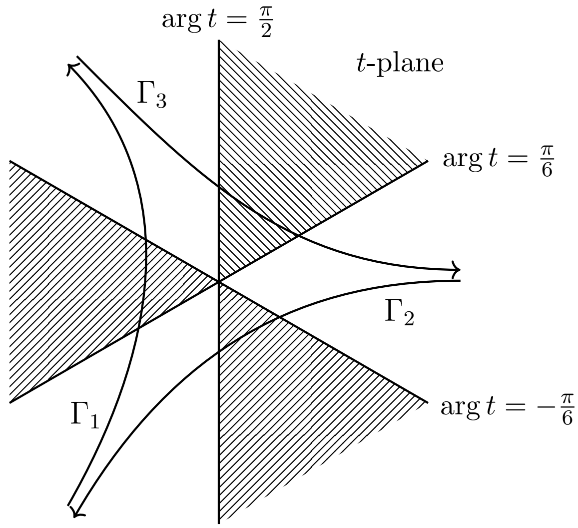


Figure 3: [58, Figure 22.1]: Integration contours Γ_1, Γ_2, Γ_3.

Example 6.1 (Airy)**.** The Fourier–Laplace transform of $P_{\text{Airy}} = \partial^2 - x$ is

$$\mathcal{F}\big\{P_{\text{Airy}}\big\} \;=\; t^2 + \partial_t \;\in\; \mathbb{C}[t]\langle\partial_t\rangle\,.$$

The solution space of $\mathcal{F}\{P_{\text{Airy}}\}$ is spanned by the exponential function $\exp(-t^3/3)$. The solutions of Airy's equation are then obtained by taking the inverse Fourier–Laplace transform of $\exp(-t^3/3)$, i.e., integrals of the form

$$\mathcal{F}^{-1}\big\{e^{-t^3/3}\big\} \;=\; \int_{\Gamma_i} e^{-t^3/3} e^{xt}\,\mathrm{d}t\,. \tag{6.3}$$

Figure 3 shows three integration contours $\Gamma_1, \Gamma_2, \Gamma_3$ so that the integral in (6.3) converges. Only two of the three are linearly independent: their composition can be contracted to a point on the Riemann sphere. The integrals over two such contours span the two-dimensional solution space of Airy's equation (1.2). $\diamond$

Via the isomorphism (6.2), we now define the Fourier–Laplace transform for D-modules. Let M be a D-module. Its Fourier–Laplace transform $\mathcal{F}\{M\}$ is the following module over the Weyl algebra $D_t = \mathbb{C}[t_1,\ldots,t_n]\langle\partial_{t_1},\ldots,\partial_{t_n}\rangle$ in the t-variables. It is the same abelian group, with the action of D_t induced by the isomorphism (6.2), i.e., for $m \in M$:

$$t_i \bullet m \;=\; -\partial_i \bullet m \quad \text{and} \quad \partial_{t_i} \bullet m \;=\; x_i \bullet m\,. \tag{6.4}$$

The Fourier–Laplace transform of a holonomic D-module is holonomic again, but its holonomic rank typically changes. In the recent work [38], the authors describe the enhanced solution complex of the Fourier–Laplace transform of holonomic D-modules in dimension one. In particular, they construct natural bases of the space of holomorphic solutions in terms of rapid decay homology.

A further integral transform that is in common use is the Mellin transform, which is for instance used to construct shift relations among master integrals, see [1, Section 3] for details.

Definition 6.2. Let f be a function of n variables. The *Mellin transform* of f is defined to be the function in the variables $\nu = (\nu_1, \ldots, \nu_n)$ given by

$$\mathfrak{M}\{f\}(\nu) \coloneqq \int_\Gamma f\, x^\nu \, \frac{\mathrm{d}x}{x}\,, \tag{6.5}$$

where $x^\nu \frac{\mathrm{d}x}{x}$ denotes the n-form $x_1^{\nu_1} \cdots x_n^{\nu_n} \frac{\mathrm{d}x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}x_n}{x_n}$ and Γ is an appropriately chosen integration contour.

Example 6.3. Let $f(x) = \exp(-x)$. Its Mellin transform is the gamma function:

$$\mathfrak{M}\{e^{-x}\}(\nu) \;=\; \Gamma(\nu) \;=\; \int_0^\infty e^{-x} x^{\nu-1}\, \mathrm{d}x\,.$$

For $\nu \in \mathbb{N}_{>0}$, $\Gamma(\nu) = (\nu-1)!$ is the factorial. ◇

Remark 6.4. Classically, the integration contour $\Gamma = \mathbb{R}^n_{>0}$ in (6.5) is the positive orthant in $\mathbb{R}^n$. However, this imposes strong conditions on f for the integral (6.5) to converge. One may instead adapt the integration contour Γ to the integrand: *twisted* cycles $\Gamma \in H_n((\mathbb{C}^*)^n \setminus V(f), \operatorname{dlog}(f x^\nu))$ ensure the convergence of the integral. Here, dlog denotes the logarithmic differential, i.e., $\operatorname{dlog}(f x^\nu)$ denotes the differential one-form

$$\operatorname{dlog}(f x^\nu) \;=\; \frac{1}{f} \sum_{i=1}^n \frac{\partial f}{\partial x_i} \mathrm{d}x_i + \sum_{i=1}^n \nu_i \, \frac{\mathrm{d}x_i}{x_i}$$

with poles along $V(f)$. The dependency on ν is discussed in detail in [1, Section 3]. ◇

The Mellin transform obeys the following rules:

$$\mathfrak{M}\{x_i \cdot f\}(\nu) \;=\; \mathfrak{M}\{f\}(\nu + e_i)\,, \quad \mathfrak{M}\left\{x_i \cdot \frac{\partial f}{\partial x_i}\right\}(\nu) \;=\; -\nu_i \cdot \mathfrak{M}\{f\}(\nu)\,. \tag{6.6}$$

The Mellin transform $\mathfrak{M}\{\cdot\}$ hence turns multiplication by $x_i^{\pm 1}$ into shifting the new variable ν_i by ± 1, and the action of the ith Euler operator $\theta_i = x_i \partial_i$ into multiplication by $-\nu_i$.

Definition 6.5. The (n-th) *shift* (or *difference*) *algebra* with polynomial coefficients

$$S_n \;\coloneqq\; \mathbb{C}[\nu_1, \ldots, \nu_n]\langle \sigma_1^{\pm 1}, \ldots, \sigma_n^{\pm 1} \rangle$$

is obtained from the free $\mathbb{C}$-algebra generated by $\{\nu_i, \sigma_i, \sigma_i^{-1}\}_{i=1,\ldots,n}$ by imposing the following relations: all generators commute, except ν_i and the shift-operators σ_i. They obey the rule

$$\sigma_i^{\pm 1} \nu_i \;=\; (\nu_i \pm 1) \sigma_i^{\pm 1}.$$

This implies that $\sigma^a \nu^b = (\nu + a)^b \sigma^a$ for any $a \in \mathbb{Z}^n$, $b \in \mathbb{N}^n$. The shift algebra naturally comes into play when studying the Mellin transform of functions. There is a natural action of S_n on the Mellin transform of functions: it shifts the variable ν_i by 1, i.e.,

$$\sigma_i \bullet \mathfrak{M}\{f\}(\nu) \;=\; \mathfrak{M}\{f\}(\nu + e_i)\,,$$

which justifies the name "shift operator" and also explains the rule in (6.6). Mimicking the rules in (6.6), the *(algebraic) Mellin transform* of [43] is the isomorphism of $\mathbb{C}$-algebras

$$\mathfrak{M}\{\cdot\}\colon D_{\mathbb{G}_m^n} \longrightarrow S_n\,, \quad x_i^{\pm 1} \mapsto \sigma_i^{\pm 1},\;\; \theta_i \mapsto -\nu_i\,. \tag{6.7}$$

The notation $\mathfrak{M}\{\cdot\}$ is used both for the Mellin transform of *functions* and that of *operators*. This is justified by the fact that $\mathfrak{M}\{\cdot\}$ is compatible with the action of the $D_{\mathbb{G}_m^n}$ and S_n, i.e.,

$$\mathfrak{M}\{P \bullet f\} \;=\; \mathfrak{M}\{P\} \bullet \mathfrak{M}\{f\}\,.$$

Example 6.6 (Example 6.3 revisited)**.** The function $f(x) = \exp(-x)$ is annihilated by $P = \partial + 1$. Its Mellin transform is

$$\mathfrak{M}\{P\} \;=\; \mathfrak{M}\left\{\frac{1}{x}x\partial + 1\right\} \;=\; -\sigma^{-1}\nu + 1 \;=\; -(\nu-1)\sigma^{-1} + 1\,.$$

From $P \bullet f = 0$, we conclude that $\mathfrak{M}\{P\} \bullet \mathfrak{M}\{f\} = 0$. Writing this out yields

$$\Gamma(\nu) \;=\; (\nu-1)\cdot\Gamma(\nu-1)\,,$$

a shift relation that one is familiar with from factorials. $\diamond$

Exercise 6.7. Let $P = x_1^2\partial_1\partial_2 + \theta_2 \in D_2$. Compute its Mellin transform $\mathfrak{M}\{P\} \in S_2$. $\diamond$

6.2 Restricting and integrating

Proposition 6.8. *Let f be a holonomic function in n variables and $m < n$. Then the restriction of f to the coordinate subspace $\{x_{m+1} = \cdots = x_n = 0\}$ is a holonomic function of the variables $x_1, \ldots, x_m$.*

In the notation of the proposition, we will denote by D_m the Weyl algebra $\mathbb{C}[x_1,\ldots,x_m]\langle\partial_1,\ldots,\partial_m\rangle$ in the first m variables.

Proof. For $i \in \{m+1,\ldots,n\}$, consider the right D_n-ideal x_iD_n. This ideal is a left module over $D_m = \mathbb{C}\left[x_1,\ldots,x_m\right]\langle\partial_1,\ldots,\partial_m\rangle$. The sum of these ideals with $\operatorname{Ann}_{D_n}(f)$ is hence a left D_m-module. By [50, Proposition 5.2.4], its intersection with D_m is

$$\left(\operatorname{Ann}_{D_n}(f) + x_{m+1}D_n + \cdots + x_nD_n\right) \cap D_m\,.$$

is a holonomic D_m-ideal and it annihilates the restricted function $f(x_1,\ldots,x_m,0,\ldots,0)$. $\square$

Definition 6.9. Let I be a D_n-ideal. The D_m-ideal

$$\left(I + x_{m+1}D_n + \cdots + x_nD_n\right) \;\cap\; D_m \tag{6.8}$$

is the *restriction ideal* of I to the coordinate subspace $\{x_{m+1} = \cdots = x_n = 0\} \subset \mathbb{C}^n$.

Example 6.10 (Example 2.28 revisited)**.** The restriction ideal of $I = \langle\partial_1x_1\partial_1, \partial_2^2+1\rangle$ to its singular locus $\operatorname{Sing}(I) = \{x_1 = 0\}$ is the $\mathbb{C}[x_2]\langle\partial_2\rangle$-ideal $\langle\partial_2^2+1\rangle$, which can be computed by running the following lines in Singular with the library `dmodapp_lib`.

```
LIB "dmodapp.lib";
def D2 = makeWeyl(2); setring D2;
ideal I = D(1)*x(1)*D(1), D(2)^2+1;
intvec w = 1,0; def Ires = restrictionIdeal(I,w);
setring Ires; resIdeal;
```

The restriction ideal has holonomic rank 2. Its solution space is $\mathbb{C} \cdot \{\cos(x_2), \sin(x_2)\}$. $\diamond$

Example 6.11. Let $f(x_1, x_2) = (x_1 + x_2)^3$. Since f is symmetric in its variables and homogeneous of degree 3, it is annihilated by the hypergeometric D-ideal

$$I = \left\langle \theta_1 + \theta_2 - 3,\, \partial_1 - \partial_2 \right\rangle .$$

The restriction of I to $\{x_2 = 0\}$ is the $\mathbb{C}[x_1]\langle\partial_1\rangle$-ideal $\langle \theta_1 - 3, \partial_1^4 \rangle$. One checks that, indeed, $f(x_1, 0)$ is annihilated by it. $\diamond$

The computation of restriction ideals is hard and terminates only for small examples. To compute restriction ideals, the authors of [8] take a detour via Pfaffian systems; the latter will be topic of Section 6. Their implementations are made available in the open source computer algebra system `Risa/Asir` [45].

Definition 6.12. Let I be a D_n-ideal. The D_m-ideal

$$\big(I + \partial_{m+1} D_n + \cdots + \partial_n D_n\big) \cap D_m \qquad \text{for } \; m < n \tag{6.9}$$

is called the *integration ideal* of I with respect to the variables $x_{m+1}, \ldots, x_n$.

One technique to compute integration D-ideals is via "creative telescoping" [36]. Latest algorithms for multivariate integration D-ideals are provided in [6].

The expression in Equation (6.9) is dual to the restriction ideal (6.8) under the Fourier transform (6.2). If $I = \operatorname{Ann}(f)$ for a holonomic function $f \colon \mathbb{R}^n \to \mathbb{C}$, the definite integral

$$F(x_1, \ldots, x_{n-1}) = \int_a^b f(x_1, \ldots, x_{n-1}, x_n)\, \mathrm{d}x_n$$

is a holonomic function in $m = n-1$ variables, assuming the integral exists, and is annihilated by the integration ideal (6.9); see [52, Proposition 2.11] for a detailed discussion of how to take the boundary terms a, b of the integration domain into account.

Example 6.13. Consider the D_2-ideal $I = \big\langle \theta_1 + 1, \theta_2 + 1 \big\rangle$. It has holonomic rank 1 and its solution space is spanned by the function $f(x_1, x_2) = \frac{1}{x_1 x_2}$. The integration ideal of I with respect to the variable x_2 is the D_1-ideal $(I + \partial_2 D_2) \cap D_1 = \langle \theta_1 + 1 \rangle$. Indeed, it annihilates the integral $\int_a^b f(x_1, x_2)\, \mathrm{d}x_2 = (\ln|b| - \ln|a|) \cdot \frac{1}{x_1}$. $\diamond$

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