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Algebraic Computations with Linear PDEs behind Functions in the Sciences

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Encoding observables by **linear** differential equations

A warm-up example

$y''(x) = -y(x)$ together with $y(0) = 0,\ y'(0) = 1$ encodes the sine function.

Supernumerary rainbows

Airy's equation: $f''(x) = x \cdot f(x)$

Stokes' phenomenon

Picture: Supernumerary rainbows over Berlin in April, 2020.
Credits: Taken by Johannes Bahrdt, published on Wikipedia under Creative Commons license.

A vibrating ukulele string

The displacement u of an oscillating string obeys the 1-dimensional wave equation:

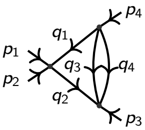
$$\frac{\partial^2 u}{\partial t^2} = c \cdot \frac{\partial^2 u}{\partial x^2}$$

x position, t time

$$c^2 = \frac{\text{tension of the string}}{\text{density of the string}}$$

Picture of my ukulele: taken by myself.

Differential equations behind holonomic functions



solution to a maximally overdetermined system of PDEs

$f(x_1, \ldots, x_n)$ a holonomic function $x = (x_1, \ldots, x_n)$

E.g., various special functions, Feynman integrals, some likelihood functions, ...

Linear PDEs fulfilled by f occur in two shapes:

polynomial coefficients

1. holonomic system of linear PDEs with polynomial coefficients

$$\sum_{\substack{\alpha,\beta\in\mathbb{N}^n,\\ \text{finite}}} c_{\alpha\beta} x^\alpha f^{(\beta_1,\ldots,\beta_n)}(x) = 0$$

2. in matrix form $\boxed{dF = A \cdot F}$

   A a matrix of differential one-forms

   F a vector of functions

Different advantages:

over $\mathbb{C}(x_1, \ldots, x_n)$

1. makes properties of functions manifest
2. linear algebra toolbox directly available

Systematic computation of connection matrices with tools from

algebraic analysis, Gröbner basis theory, computer algebra software.

D-modules symbolic computations

[1] P. Görlach, J. Koefler, A.-L.S., M. Sayrafi, H. Schroeder, N. Weiss, and F. Zaffalon. Connection Matrices in *Macaulay2*. Preprint `arXiv:2504.01362`. `ConnectionMatrices.m2`

[2] D. R. Grayson and M. E. Stillman. *Macaulay2*, a software system for research in algebraic geometry. Available at `https://macaulay2.com/`. *open source!*

Non-commutative algebras of differential operators

f a function of $x = (x_1, \ldots, x_n)$

From homogeneous, linear differential equations to differential operators:

The PDE $\sum_{\alpha,\beta\in\mathbb{N}^n} c_{\alpha,\beta} x^\alpha f^{(\beta_1,\ldots,\beta_n)}(x) = 0$ is encoded by $P = \sum_{\alpha,\beta\in\mathbb{N}^n} c_{\alpha,\beta} x^\alpha \partial^\beta$.

$P \bullet f = 0, \; \partial_i \bullet f = \frac{\partial f}{\partial x_i}$

Example: $f''(x) - x \cdot f(x) = 0$ is encoded by $P_{\text{Airy}} = \partial^2 - x$.

ODE ($n = 1$)

Weyl algebra: $D_n = \mathbb{C}[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n \rangle$

All generators commute, except ∂_i, x_i. They obey the Leibniz rule: $\overbrace{\partial_i x_i - x_i \partial_i}^{=[\partial_i, x_i]} = 1$.

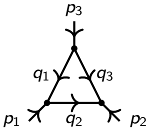
⚠ Left D-ideals encode crucial properties of their solutions. $I = \langle P_1, \ldots, P_k \rangle \subset D_n$
such as singularities, growth behavior, number of data needed to encode the function, ...

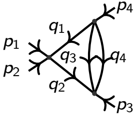
Examples:

- The **annihilator** of f is $\mathrm{Ann}_D(f) := \{ P \in D_n \mid P \bullet f = 0 \}$. *a left D_n-ideal*
  E.g., $\mathrm{Ann}_D(x^n) = \langle x\partial - n, \partial^{n+1} \rangle \subset D_1$.
- If $x_1\partial_1 + \cdots + x_n\partial_n - k \in \mathrm{Ann}_D(f)$, then f is homogeneous of degree k.
  I.e., $f(\lambda x_1, \ldots, \lambda x_k) = \lambda^k \cdot f(x_1, \ldots, x_n)$ for all $\lambda \neq 0$ and all x.
- Invariance under the conformal group is encoded as being a solution to a D-ideal.
  ⇝ Ward identities

$$\mathcal{A} = \sum_G a_G \mathcal{I}_G$$

$\mathcal{A}\colon \left(\mathbb{R}^{1,d-1}\right)^n \longrightarrow \mathbb{C}$ the amplitude of a scattering process of n particles

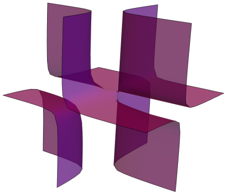




Feynman diagram: $G = (V, E)$ + "external legs"

- external legs labeled with momentum vectors $p_i \in \mathbb{R}^{1,d-1}$
- each $e_i \in E$ equipped with a momentum vector $q_i \in \mathbb{R}^{1,d-1}$ and a mass $m_i \in \mathbb{R}_{\geq 0}$

Symanzik polynomials: $\mathcal{U}_G, \mathcal{F}_G \in \mathbb{R}[\{\alpha_i\}_{e_i \in E}]$
in "Schwinger parameters" α_i



$\mathcal{G}_G = \mathcal{U}_G + \mathcal{F}_G$ **graph polynomial** of G

Example (parachute): $\mathcal{U}_G = (\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4) + \alpha_3\alpha_4$

$\mathcal{F}_G = (p_1 + p_2)^2 \alpha_1\alpha_2(\alpha_3 + \alpha_4) + p_3^2 \alpha_2\alpha_3\alpha_4 + p_4^2 \alpha_1\alpha_3\alpha_4 - \mathcal{U}_G \cdot \left(\sum_{i=1}^4 m_i^2 \alpha_i\right)$

Feynman integral (Lee–Pomeransky representation): $\ell > 1$

$$\mathcal{I}_G = N_\nu \cdot \int_{\mathbb{R}^n_{>0}} \left(\mathcal{G}_G\right)^{-d/2} \alpha_1^{\nu_1} \cdots \alpha_n^{\nu_n} \frac{\mathrm{d}\alpha_1}{\alpha_1} \wedge \cdots \wedge \frac{\mathrm{d}\alpha_n}{\alpha_n}$$

ν's: kinematic parameters
N_ν a scaling factor
dim. regularization: $d = 4 - 2\varepsilon$

[3] S. Weinzierl. *Feynman Integrals: A Comprehensive Treatment for Students and Researchers*. In *UNITEXT for Physics*, Springer, 2022.
[4] C. Fevola and A.-L.S. Algebraic and Positive Geometry of the Universe: from Particles to Galaxies. *Notices of the American Mathematical Society*, 72(8):808–817, 2025.

Generalized Euler integrals as holonomic functions

$$\mathcal{I}(c) = \int_\Gamma f_1^{s_1} \cdots f_k^{s_k} x_1^{\nu_1} \cdots x_n^{\nu_n} \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$$

generalizing the β-function, ${}_2F_1$, ...

$\{A_j\}_{j=1,\ldots,k} \overset{\text{finite}}{\subset} \mathbb{Z}^n$ representing the monomial supports of Laurent polynomials $f_j = \sum_{u \in A_j} c_{u,j} \alpha^u$

$$A := \left(\begin{array}{ccc|ccc|c|ccc} & A_1 & & & A_2 & & & & A_k & \\ 1 & \cdots & 1 & 0 & \cdots & 0 & & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 & & & & \\ & & & & & & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & & 1 & \cdots & 1 \end{array}\right)$$

$$\boxed{D_A = \mathbb{C}[c_u \mid u \in A]\langle \partial_u \mid u \in A \rangle}$$

Weyl algebra in variables indexed by the columns of A

Two ingredients for GKZ systems:

- toric ideal of A: $I_A := \left\langle \partial^a - \partial^b \mid a - b \in \ker(A),\ a, b \in \mathbb{N}^A \right\rangle$
- For $\kappa \in \mathbb{C}^{n+k}$: $J_{A,\kappa}$ the D_A-ideal generated by the entries of $A\theta - \kappa$
  $\theta = (\theta_u)_{u \in A}$, $\theta_u = c_u \partial_u$ Euler operator

GKZ system of (A, κ): $\boxed{H_A(\kappa) = I_A + J_{A,\kappa} \subset D_A}$ $\quad \mathcal{I} \in \mathrm{Sol}\,(H_A(-\nu, s))$

Feynman integrals: many coefficients constrained $\quad k = 1,\ f = \mathcal{G}_G,\ s = -d/2$

⚠ need to **restrict** $H_A(\kappa)$ to the physically relevant subspace of coefficients *Challenge for PDEs!*

[5] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky. Generalized Euler integrals and A-hypergeometric functions. *Advances in Mathematics*, 84(2):255–271, 1990.
[6] L. de la Cruz. Feynman integrals as A-hypergeometric functions. *J. High Energy Phys.*, 2019(123), 2019.

Non-commutative algebras of differential operators cont'd

$\mathbb{C}(x) = \left\{ \frac{p}{q} \mid p, q \in \mathbb{C}[x],\ q \neq 0 \right\}$ the field of rational functions in $x = (x_1, \ldots, x_n)$

Rational Weyl algebra: $R_n = \mathbb{C}(x_1, \ldots, x_n)\langle \partial_1, \ldots, \partial_n \rangle.$

⚠ The differences between $I \subset D_n$ and $R_n I \subset R_n$ are subtle. The **Weyl closure** of I, $W(I) := R_n I \cap D_n$, fulfills $I \subseteq W(I) \subseteq R_n I$, $R_n I = R_n W(I)$, $\mathrm{Sing}(W(I)) \subseteq \mathrm{Sing}(I)$, $\mathrm{rank}(W(I)) = \mathrm{rank}(I)$, and $\mathrm{Sol}(I) = \mathrm{Sol}(W(I))$.

The **holonomic rank** of a D_n-ideal I is the dimension of the $\mathbb{C}(x_1, \ldots, x_n)$-vector space underlying $R_n / R_n I$:

$$\mathrm{rank}(I) = \dim_{\mathbb{C}(x_1, \ldots, x_n)} (R_n / R_n I) \qquad P \sim Q \Leftrightarrow P - Q \in R_n I$$

◇ **generically:** $\mathrm{rank}(I)$ = dimension of space of analytic solutions = size of connection matrix
◇ **for a single ODE ($n = 1$, $I = D_1 P$):** $\mathrm{rank}(I) = \mathrm{ord}(P)$

Example: $I = \langle x_1 \partial_2^2 - x_2 \partial_1^2 + \partial_1 - \partial_2, x_1 \partial_1 + x_2 \partial_2 + 1 \rangle \subset D_2$

- ◇ $\{1, \partial_1\}$ is a $\mathbb{C}(x_1, x_2)$-basis of $R_2 / R_2 I$
- ◇ $\mathrm{rank}(I) = 2$
- ◇ $\mathrm{Sing}(I) = V(x_1^3 - x_2^3) \subset \mathbb{C}^2$ singular locus of I

[7] H. Tsai. Weyl closure of a linear differential operator. J. Symbolic Comput., 29:747–775, 2000

Writing D-ideals in matrix form

Let $I \subset D_n$ with $\operatorname{rank}(I) = m$ and $\{s_1 = 1, s_2, \ldots, s_m\}$ a $\mathbb{C}(x_1, \ldots, x_n)$-basis of R_n/R_nI. For $f \in \operatorname{Sol}(I)$ a solution to I, denote $F = (f, s_2 \bullet f, \ldots, s_m \bullet f)^\top$. Then there exist unique matrices $A_1, \ldots, A_n \in \operatorname{Mat}_{m\times m}(\mathbb{C}(x_1, \ldots, x_n))$ such that

$$\boxed{\partial_i \bullet F \;=\; A_i \cdot F} \qquad \text{Pfaffian system of } I$$

for any $f \in \operatorname{Sol}(I)$. The **connection matrices** A_i fulfill the integrability conditions:

$$\boxed{\partial_i \bullet A_j - \partial_j \bullet A_i \;=\; [A_i, A_j]} \qquad \text{for all } \; i,j = 1, \ldots, n.$$

Gauge transformation: For $\widetilde{F} = BF$ with $B \in \operatorname{GL}_m(\mathbb{C}(x_1, \ldots, x_n))$, the transformed system is $\partial_i \bullet \widetilde{F} = \widetilde{A}_i \widetilde{F}$ with $\boxed{\widetilde{A}_i = BA_iB^{-1} + (\partial_i \bullet B) \cdot B^{-1}.}$

Facts about connection matrices:

- geometrically, they arise from a vector bundle with an integrable connection
- systematic computation via Gröbner bases in R_n
- dually, the connection matrix is $\boxed{A \;=\; A_1 \mathrm{d}x_1 + \cdots + A_n \mathrm{d}x_n}$

Particularly popular in dimensional regularization: ε-factorized form $\boxed{\mathrm{d}F = \varepsilon \widetilde{A} F}$

Then: $D_n(\varepsilon) = \mathbb{C}(\varepsilon)[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n \rangle$ $\qquad$ $\widetilde{A}$ independent of ε

A friendly example

$$I \;=\; \left\langle x_1\partial_2^2 - x_2\partial_1^2 + \partial_1 - \partial_2,\; x_1\partial_1 + x_2\partial_2 + 1 \right\rangle \;\subset\; D_2 = \mathbb{C}[x_1,x_2]\langle \partial_1, \partial_2\rangle$$

- ◇ $\{1, \partial_1\}$ is a $\mathbb{C}(x_1,x_2)$-basis of $R_2/R_2 I$ $\quad$ $\operatorname{rank}(I) = 2$
- ◇ $\operatorname{Sol}(I) = \mathbb{C} \cdot \left\{ \frac{1}{x_1-x_2}, \frac{1}{x_1-x_2}\log(x_1/x_2) \right\}$
- ◇ $\operatorname{Sing}(I) = V\left(x_1^3 - {x_2}^3\right) = V\left((x_1-x_2)\left(x_1^2 + x_1x_2 + {x_2}^2\right)\right) \subset \mathbb{C}^2$

Denoting $\boxed{F = (f, \partial_1 \bullet f)^\top = \begin{pmatrix} f \\ \frac{\partial f}{\partial x_1} \end{pmatrix}}$ for $f \in \operatorname{Sol}(I)$,

$$\partial_1 \bullet F \;=\; \underbrace{\begin{pmatrix} 0 & 1 \\ -\frac{1}{x_1(x_1-x_2)} & \frac{-3x_1+x_2}{x_1(x_1-x_2)} \end{pmatrix}}_{=A_1} \cdot F$$

and

$$\partial_2 \bullet F \;=\; \underbrace{\begin{pmatrix} -\frac{1}{x_2} & -\frac{x_1}{x_2} \\ \frac{1}{(x_1-x_2)x_2} & \frac{x+x_2}{(x_1-x_2)x_2} \end{pmatrix}}_{=A_2} \cdot F.$$

Dually:

$$\mathrm{d}F \;=\; \underbrace{\begin{pmatrix} -\frac{1}{x_2}\mathrm{d}x_2 & \mathrm{d}x_1 - \frac{x_1}{x_2}\mathrm{d}x_2 \\ -\frac{1}{x_1(x_1-x_2)}\mathrm{d}x_1 + \frac{1}{(x_1-x_2)x_2}\mathrm{d}x_2 & -\frac{3x_1-x_2}{x_1(x_1-x_2)}\mathrm{d}x_1 + \frac{x_1+x_2}{(x_1-x_2)x_2}\mathrm{d}x_2 \end{pmatrix}}_{=A} \cdot F\,.$$

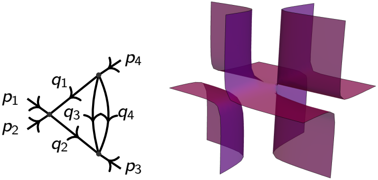
$\rightsquigarrow$ Systematic computation with the help of **Gröbner bases** in Weyl algebras!

Gröbner bases in algebraic geometry

Commutative setup

- ideal $I \subset \mathbb{C}[x_1, \dots, x_n]$ in a polynomial ring
- **algebraic variety** $V(I) \subset \mathbb{C}^n$: common vanishing set of all $f \in I$

Example: graph polynomial of a Feynman diagram G, $\mathcal{G}_G = \mathcal{F}_G + \mathcal{U}_G \in \mathbb{C}[\{\alpha_i\}]$ in Schwinger parameters α_i.



The set of real zeros of $\mathcal{G}_G = \mathcal{U}_G + \mathcal{F}_G$ for G the massless parachute diagram.

Ideals can be represented in various different ways!

$$I = \langle p_1, \dots, p_k \rangle = \langle q_1, \dots, q_\ell \rangle = \dots$$

In the Weyl algebra D_4: $\langle \partial_1, \partial_2, \partial_3, \partial_4 \rangle =$

$= \langle \partial_1, x_1^4 x_4 \partial_4 + x_1^2 x_3 \partial_3 + x_1^3 \partial_4 + x_1 \partial_3 + \partial_2 \rangle$

Gröbner bases of ideals

- particular generating set, depending on choice of a term order on $\mathbb{C}[x_1, \dots, x_n]$

  e.g., lexicographic order built on $x_1 \succ \dots \succ x_n$
- comparability and properties of ideals
- computations: solving polynomial systems, geometric operations, ...

[8] D. Cox, J. Little, and D. O'Shea. *Ideals, Varieties and Algorithms. An Introduction to Computational Algebraic Geometry and Commutative Algebra.* Undergraduate Texts in Mathematics, Springer, 1997.

Every $P \in D_n$ can be written as $P = \sum_{\alpha,\beta\in\mathbb{N}^n} c_{\alpha,\beta} x^\alpha \partial^\beta.$ *finitely many* $c_{\alpha,\beta} \neq 0$

A **term order** on D_n is a total order $\prec$ on the set of monomials $\{x^\alpha\partial^\beta\}$ in D_n.
+ *technical assumptions adapting to non-commutativity*

Example: $\prec_{(0,v)}$ with $v \in \mathbb{R}^n_{>0}$ and $\prec$ the lexicographic order built upon $\partial_1 \succ \cdots \succ \partial_n \succ x_1 \succ \cdots \succ x_n$. *first compare* $(0,v)$*-weight,* `lex` *as a tie breaker*
This defines an **elimination order** on D_n, i.e., $\partial^\beta \prec \partial^\gamma$ implies $x^\alpha\partial^\beta \prec \partial^\gamma \ \forall \alpha \in \mathbb{N}^n$.

Gröbnes bases of D_n-ideals

- ◇ The **initial monomial** of $P \in D_n$, denoted $\mathrm{in}_\prec(P)$, is the monomial $x^\alpha\partial^\beta$ in $\mathbb{C}[x_1,\ldots,x_n,\partial_1,\ldots,\partial_n]$ for which $x^\alpha\partial^\beta$ is the largest monomial of P.
- ◇ The **initial ideal** of $I \subset D_n$, $\mathrm{in}_\prec(I)$, is the ideal in $\mathbb{C}[x,\partial]$ generated by $\{\mathrm{in}_\prec(P) \mid P \in D_n\}$.
- ◇ A finite set $G = \{G_1,\ldots,G_\ell\} \subset D_n$ is a **Gröbner basis** of I w.r.t. $\prec$ if $I = D_n G$ and $\mathrm{in}_\prec(I)$ is generated by $\{\mathrm{in}_\prec(G_i) \mid G_i \in G\}$.
- ◇ The **standard monomials** of I with respect to $\prec$ are the monomials $x^\alpha\partial^\beta \notin \mathrm{in}_\prec(I)$.

Example: $I = \left\langle x_1\partial_1^2 - \underline{x_2\partial_2^2} + \partial_1 - \partial_2,\ x_1\partial_1 + \underline{x_2\partial_2} + 1 \right\rangle \subset D_2$

A Gröbner basis of I with respect to $\prec_{(0,v)}$ for $v = (1,2)$ is

$$\left\{ \underline{x_2\partial_2} + x_1\partial_1 + 1,\ \underline{x_1^2\partial_1^2} - x_1x_2\partial_1^2 + 3x_1\partial_1 - x_2\partial_1 + 1,\ \underline{x_1\partial_1\partial_2} + x_1\partial_1^2 + \partial_2 + \partial_1 \right\}.$$

Computation of connection matrices

$R_n = \mathbb{C}(x_1, \ldots, x_n)\langle\partial_1, \ldots, \partial_n\rangle$ $\mathbb{C}(x)$ *now considered as coefficients*

Restricting a term order $\prec$ on D_n to monomials in the ∂_i's yields a term order $\prec'$ on R_n.

Nota bene: The standard monomials of a Gröbner basis of $R_n I$ are a $\mathbb{C}(x_1, \ldots, x_n)$-basis of $R_n / R_n I$. *Macaulay's Basis Theorem*

Entries of the connection matrices

Let G be a Gröbner basis of $R_n I$ with respect to a term order $\prec'$ on R_n with standard monomials $\{s_1 = 1, s_2, \ldots, s_m\}$. Therefore, one can write

$$\partial_i s_j = \sum_{k=1}^{m} a_{jk}^{(i)} s_k + Q_j^{(i)}$$

with $Q_j^{(i)} \in R_n I$.

$F = (f, s_2 \bullet f, \ldots, s_m \bullet f)^\top$
$\partial_i \bullet F = A_i \cdot F$ for any $f \in \mathrm{Sol}(I)$

Hence $a_{jk}^{(i)}$ is the (j,k)-th entry of the matrix A_i. *reduction in R_n*

Main step: normal form algorithm in R_n $\rightsquigarrow$ `normalForm`$(\partial_i s_j, G)$ yields A_i

⚠ The rational Weyl algebra R_n is not implemented in *Macaulay2*.

Workaround: If G is a Gröbner basis of a D_n-ideal I w.r.t. an elimination order $\prec$ on D_n, then G is also a Gröbner basis of $R_n I \subset R_n$ with respect to the order $\prec'$ on R_n.

[9] M. Saito, B. Sturmfels, and N. Takayama. *Gröbner deformations of hypergeometric differential equations.* Volume 6 of *Algorithms and Computation in Mathematics*. Springer, 2000.

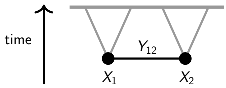
Example code in *Macaulay2*

Package & documentation available at: `https://mathrepo.mis.mpg.de/ConnectionMatrices`

Try it out on your favorite system of PDEs!

```
i1 : needsPackage "ConnectionMatrices";
i2 : D = makeWeylAlgebra(QQ[x,y],{1,2});
i3 : I = ideal(x*dx^2-y*dy^2+dx-dy,x*dx+y*dy+1);
i4 : SM = standardMonomials I
o4 = {1, dx}
i5 : A = connectionMatrices(I,SM)
o5 = {| 0              1                |, | (-1)/y     (-x)/y         |}
      | (-1)/(x2-xy)  (-3x+y)/(x2-xy)  |   | 1/(xy-y2)  (x+y)/(xy-y2)  |
i6 : connectionMatrix I
o6 = | (-1)/ydy                          (-x)/ydy+dx                          |
     | 1/(xy-y2)dy+(-1)/(x2-xy)dx  (x+y)/(xy-y2)dy+(-3x+y)/(x2-xy)dx |
```

More interesting examples provided in the documentation:



- ◇ Gauss' hypergeometric function ${}_2F_1$
- ◇ massless one-loop Feynman triangle integral
- ◇ cosmological correlator for 2-site chain *+ gauge transformation to ε-factorized form*

[10] J. M. Henn, E. Pratt, A.-L.S., and S. Zoia. *D*-Module Techniques for Solving Differential Equations in the Context of Feynman Integrals. *Letters in Mathematical Physics*, 114(87), 2024.

[11] C. Fevola, G. L. Pimentel, A.-L.S., and T. Westerdijk. Algebraic Approaches to Cosmological Integrals. Special volume on Positive Geometry, *Le Matematiche*, 80(1):303–324, 2025.

Conclusion

Algebraic analysis provides tools for the systematic investigation and manipulation of systems of PDEs behind holonomic functions in the sciences.

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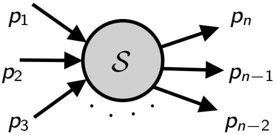
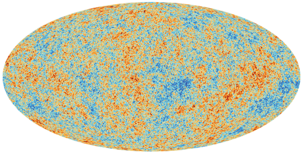
The universe on small and on large scale



A graphical representation of a scattering process between n elementary particles, as happens e.g. in a particle accelerator.



The cosmic microwave background (CMB) is a remnant of the first light in our universe. *Credit: ESA, Planck Collaboration, 2013.*

Mathematical object of study: close bond to generalized Euler integrals

$$\mathcal{I}(c) = \int_\Gamma f_1^{s_1} \cdots f_k^{s_k} x_1^{\nu_1} \cdots x_n^{\nu_n} \frac{\mathrm{d}x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}x_n}{x_n}$$

Tools: algebraic geometry, algebraic analysis, combinatorial algebraic geometry
. . . all contributing to shaping the flourishing field of **positive geometry**

[12] K. Ranestad, B. Sturmfels, and S. Telen. What is Positive Geometry? Special volume on Positive Geometry, *Le Matematiche*, 80(1):3–16, 2025.