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IN DEN NATURWISSENSCHAFTEN

Border Bases in the Rational Weyl Algebra

Based on joint work with Carlos Rodriguez.

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March 24, 2026

Zero-dimensional ideals in polynomial rings

Let $S = \mathbb{C}[X_1, \ldots, X_n]$. The **Hilbert scheme of m points in affine n-space**,

$$\boxed{\mathrm{Hilb}_n^m = \{I \subset S \mid \dim_{\mathbb{C}}(S/I) = m\}},$$

classifies ideals $I \subset S$ whose quotient ring is m-dimensional as a $\mathbb{C}$-vector space.

Properties of Hilb_n^m

- ◇ It is connected. Hartshorne
- ◇ It can be covered by open affine subschemes...
  - ▶ ...with a combinatorial description in terms of partitions of integers.
  - ▶ ...whose ideals are naturally tied to border bases

Two questions

1. Can this be transferred to the Weyl algebra?
2. Is it a useful thing to do?

Answers in [3]: Yes!

[1] E. Miller and B. Sturmfels. *Combinatorial Comm. Algebra*, volume 227 of *Univ. Lecture Ser.*, Springer, 2005.
[2] M. Kreuzer and L. Robbiano. The geometry of border bases. *J. Pure Appl. Algebra*, 215:2005–2018, 2011.
[3] C. Rodriguez and ALS. Border Bases in the Rational Weyl Algebra. *Adv. Appl Math.*, 177:103065, 2026.

A motivating example from the commutative case: 3 points in the plane

The Hilbert scheme Hilb_2^3 classifies $I \subset \mathbb{C}[X,Y]$ such that $\dim_{\mathbb{C}}(\mathbb{C}[X,Y]/I) = 3$.

- Hilb_2^3 can be covered by three open affine subschemes corresponding to monomial bases $(Y^2, Y, 1)$, $(X, Y, 1)$, and $(X^2, X, 1)$.
- Each $I \in \mathrm{Hilb}_2^3$ in the second chart can be represented as

$$\boxed{I = \left\langle \underline{X^2} - aX - bY - c,\ \underline{XY} - dX - eY - f,\ \underline{Y^2} - gX - hy - i \right\rangle.} \tag{1}$$

- In the $\mathbb{C}$-basis $(X, Y, 1)$ of $\mathbb{C}[X,Y]/I$, the maps $X\cdot, Y\cdot \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}[X,Y]/I)$ are

$$M_X = \begin{pmatrix} a & d & 1 \\ b & e & 0 \\ c & f & 0 \end{pmatrix} \quad \text{and} \quad M_Y = \begin{pmatrix} d & g & 0 \\ e & h & 1 \\ f & i & 0 \end{pmatrix}.$$

- In order to be a Gröbner basis, all S-pairs of the generators of I in (1) need to reduce to zero. This is equivalent to requiring that $\boxed{M_X M_Y = M_Y M_X.}$
- M_X and M_Y commute if and only if

$$f = bg - de\,, \quad c = -ae + bd - bh + e^2\,, \quad \text{and} \quad i = -ag + d^2 - dh + eg\,.$$

This yields a 6-dimensional affine open subscheme of Hilb_2^3. denoted U_{2+1} in [1]

[1] E. Miller and B. Sturmfels. *Combinatorial Comm. Algebra*, volume 227 of *Univ. Lecture Ser.*, Springer, 2005.

One ingredient of border bases: order ideals

An **order ideal** is a non-empty subset $\lambda \subset \mathbb{N}^n$ such that

$$u \in \lambda,\ v \in \mathbb{N}^n, v \le u \implies v \in \lambda.$$ $\le$ component-wise

Equivalently: $\boxed{\mathcal{O}_\lambda = \{X^u | u \in \lambda\} \subset \mathbb{T}_n}$ $\mathbb{T}_n$ the set of monomials in $X_1, \ldots, X_n$

Example: For any choice of monomial order $\prec$, the set of standard monomials of a Gröbner basis of $I \subset S$ is an order ideal. $\{X^\alpha | X^\alpha \notin \mathrm{in}_\prec(I)\}$

The **border** of an order ideal $\mathcal{O}_\lambda$ is

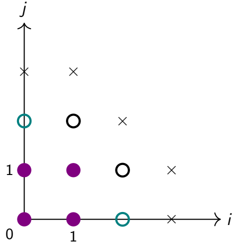
$$\boxed{\partial\mathcal{O}_\lambda = (X_1\mathcal{O}_\lambda \cup X_2\mathcal{O}_\lambda \cup \cdots \cup X_n\mathcal{O}_\lambda) \setminus \mathcal{O}_\lambda\,.} \qquad (2)$$

The **corners** of $\mathcal{O}_\lambda$ are the minimal gen's of the monomial ideal generated by $\mathbb{T}_n \setminus \mathcal{O}_\lambda$.

Example

Let $\mathcal{O} = \{1, X_1, X_2, X_1X_2\} \subset \mathbb{T}^2$.

- • $\lambda = \{(0,0), (1,0), (0,1), (1,1)\} \subset \mathbb{N}^2$
- ∘ $\partial\mathcal{O} = \{X_1^2, X_1^2X_2, X_1X_2^2, X_2^2\}$
- ∘ corners of $\mathcal{O}$
- × $\partial^2\mathcal{O} = \{X_1^3, X_1^3X_2, X_1^2X_2^2, X_2^3X_1, X_2^3\}$



Border prebases

If $\mathcal{O}_\lambda = \{t_1, t_2, \ldots, t_m\} \subset \mathbb{T}_n$ is an order ideal and $\partial\mathcal{O}_\lambda = \{b_1, b_2, \ldots, b_p\}$ its border, then an $\mathcal{O}_\lambda$-**border prebasis** is a set of polynomials $G_\lambda = \{g_1, g_2, \ldots, g_p\}$ where

$$\boxed{g_j = b_j - \sum_{i=1}^{m} c_{i,j} t_i} \qquad \text{with } c_{i,j} \in \mathbb{C}. \qquad g_j \text{ is “marked by” } b_j \tag{3}$$

Formal multiplication matrices associated to an $\mathcal{O}_\lambda$-border prebasis

The k-th column of $M_{X_i} \in \mathrm{Mat}_{m\times m}(\mathbb{C})$ is defined to be

$$\boxed{(M_{X_i})_{*k} = \begin{cases} e_r & \text{if } X_i t_k = t_r, \\ (c_{1,s}, \ldots, c_{m,s})^\top & \text{if } X_i t_k = b_s\,, \end{cases}} \tag{4}$$

where e_r denotes the r-th standard unit vector in $\mathbb{R}^m$ and the $c_{j,s}$'s are as in (3).

Terminology: The $(M_{X_i})_{*k}$ is the result of applying the division algorithm to $X_i t_k$.

[4] A. Kehrein, M. Kreuzer, and L. Robbiano. An algebraist's view on border bases. In *Solving Polynomial Equations: Foundations, Algorithms, and Applications*, volume 14 of *Algorithms Comput. Math.*, pp. 169–202. Springer, 2005.

$I \subset S = \mathbb{C}[X_1, \ldots, X_n]$	a zero-dimensional ideal
$\mathcal{O}_\lambda = \{t_1, t_2, \ldots, t_m\} \subset \mathbb{T}_n$	an order ideal
$G = \{g_1, g_2, \ldots, g_p\} \subset I$	an $\mathcal{O}_\lambda$-border prebasis

Definition

A border prebasis G is an $\mathcal{O}_\lambda$-**border basis** of I if the residue classes of $t_1, t_2, \ldots, t_m$ form a $\mathbb{C}$-vector space basis of S/I.

Example: $I = \langle X_1^2 + X_2^2 - 2, X_1X_2 - 1 \rangle$

◇ $\dim_{\mathbb{C}}(\mathbb{C}[X_1, X_2]/I) = 4$ $(-1,-1), (1,1)$ (double) ◇ no border basis for $\{1, X_1, X_2, X_1X_2\}$
◇ $\mathcal{O}_\lambda = \{1, X_1, X_2, X_2^2\}, \partial\mathcal{O}_\lambda = \{X_1^2, X_1X_2, X_2^3\}, G = \{X_1^2 + X_2^2 - 2, X_1X_2 - 1, X_2^3 + X_1 - 2X_2\}$

Properties

- ◇ Any border basis of I also generates I as an S-ideal.
- ◇ If it exists, an $\mathcal{O}_\lambda$-border basis is unique.

Characterization

Let $\mathcal{O}_\lambda = \{t_1, t_2, \ldots, t_m\}$ be an order ideal. Then an $\mathcal{O}_\lambda$-border prebasis $G = \{g_1, \ldots, g_p\}$ is an $\mathcal{O}_\lambda$-border basis of $I = \langle g_1, \ldots, g_p \rangle$ if and only if the formal multiplication matrices are pairwise commuting, i.e., if $M_{X_i}M_{X_j} = M_{X_j}M_{X_i}$ for all i, j.

$D_n = \mathbb{C}[x_1, \dots, x_n]\langle \partial_1, \dots, \partial_n \rangle$	(n-th) **Weyl algebra**	$[\partial_i, x_j] = \delta_{ij}$
$R_n = \mathbb{C}(x_1, \dots, x_n)\langle \partial_1, \dots, \partial_n \rangle$	(n-th) **rational Weyl algebra**	
$I \subset D_n$, $J \subset R_n$	left ideals	$D_n/I \in \mathrm{Mod}(D_n)$

Notation: $\bullet : D_n \times M \longrightarrow M, \; (P, m) \mapsto P \bullet m$ E.g., $\partial_i \bullet f = \frac{\partial f}{\partial x_i}$.

The **holonomic rank** of a D_n-ideal I is the dimension of the $\mathbb{C}(x_1, \dots, x_n)$-vector space underlying R_n/R_nI.

Elements of the Weyl algebra:

$$D_n = \Big\{ \sum_{\alpha,\beta \in \mathbb{N}^n, \text{ finite}} c_{\alpha,\beta} x^\alpha \partial^\beta \mid c_{\alpha,\beta} \in \mathbb{C} \Big\}$$

Principal symbols

Let $(0,1) = (0, \dots, 0, 1, \dots, 1) \in \mathbb{R}^{2n}$. The **initial form** of $P \in D$ is

$$\mathrm{in}_{(0,1)}(P) = \sum_{\{(\alpha,\beta) \mid (\alpha,\beta)\cdot(0,1) \text{ maximal}\}} c_{\alpha,\beta} x^\alpha \partial^\beta .$$

A D_n-ideal I is **holonomic** if $\dim V(\mathrm{in}_{(0,1)}) = n$. $\geq n$ by Bernstein's inequality

Fact: $I \subset D_n$ holonomic $\Rightarrow \mathrm{rank}(I) < \infty$. The reverse implication does not hold.

[5] M. Saito, B. Sturmfels, and N. Takayama. *Gröbner Deformations of Hypergeometric Differential Equations*, volume 6 of *Algorithms and Computation in Mathematics*. Springer, 2000.

System of PDEs $\leftrightarrow$ 1st order matrix equations

An **integrable connection** is a pair (M, ∇) consisting of an m-dimensional $\mathbb{C}(x_1, \ldots, x_n)$-vector space M together with an integrable connection

$$\boxed{\nabla : M \longrightarrow M \otimes \Omega^1.} \qquad (\mathbb{C}\text{-linear, Leibniz' rule, } \nabla^2 = 0)$$

Identifying $M \cong \mathbb{C}(x_1, \ldots, x_n)^m$: $\nabla = \mathrm{d} - A\wedge$, A an $m \times m$ matrix of differential 1-forms.
Geometrically: stalk of vector bundle on $\mathbb{A}^n_{\mathbb{C}}$ at the generic point.

$M \in \mathrm{Mod}(D_n)$ is **cyclic** if there exists $v \in M$ such that $D_n \bullet v = M$.
Examples: D_n/I, holonomic D_n-modules $\supset$ integrable connections

Pfaffian system of D_n-ideals (in fact of R_n-ideals $J = R_n I$)

Let $I \subset D_n$ with $\mathrm{rank}(I) = m$, $\{s_1 = 1, s_2, \ldots, s_m\}$ a $\mathbb{C}(x_1, \ldots, x_n)$-basis of $R_n/R_n I$.
There exist unique $A_1, \ldots, A_n \in \mathrm{Mat}_{m\times m}(\mathbb{C}(x_1, \ldots, x_n))$ such that

$$\boxed{\partial_i \bullet (f, s_2 \bullet f, \cdots, s_m \bullet f)^\top = A_i \cdot (f, s_2 \bullet f, \cdots, s_m \bullet f)^\top \quad \text{for } \underline{\text{all}}\ f \in \mathrm{Sol}(I)\,.}$$

The matrices A_i fulfill the integrability conditions: $[A_i, A_j] = \partial_i \bullet A_j - \partial_j \bullet A_i$ for all i, j.

Translating:* $\boxed{A = A_1 \mathrm{d}x_1 + \cdots + A_n \mathrm{d}x_n}$ * duality swept under the rug

[6] R. Hotta, K. Takeuchi, and T. Tanisaki. *D-Modules, Perverse Sheaves, and Representation Theory*, volume 236 of *Progress in Mathematics*. Birkhäuser, 2008.
[7] J. T. Stafford. Module structure of Weyl algebras. *J. London Math. Soc.*, 18:429–442, 1978.

Previously: $S = \mathbb{C}[X_1, \ldots, X_n]$, $\mathbb{T}_n = \{X^\alpha \mid \alpha \in \mathbb{N}^n\}$ monomials in the X_i's.

Now: $R_n = \mathbb{C}(x_1, \ldots, x_n)\langle \partial_1, \ldots, \partial_n \rangle$, $\mathbb{D}_n = \{\partial^\alpha \mid \alpha \in \mathbb{N}^n\}$ monomials in the ∂_i's.

$\mathbb{C}(x_1, \ldots, x_n)$ as field of coefficients

Order ideals $\mathcal{O}_\lambda$, border prebases, formal multiplication matrices M_{∂_i}, border bases: definitions straightforward to transfer.

Proposition

Let $\mathcal{O}_\lambda = \{t_1, t_2, \ldots, t_m\}$ be an order ideal, let $J \subset R_n$ with $\mathrm{rank}(J) = m$. Assume that the residue classes of the t_i's form a $\mathbb{C}(x_1, \ldots, x_n)$-vector space basis of R_n/J. Then:

1. There exists a unique $\mathcal{O}_\lambda$-border basis G of J.
2. Let $G' \subset J$ be an $\mathcal{O}_\lambda$-border prebasis. Then G' is the $\mathcal{O}_\lambda$-border basis of J.

Characterization

$$[M_{\partial_i}, M_{\partial_j}] = \partial_j \bullet M_{\partial_i} - \partial_i \bullet M_{\partial_j}$$

An $\mathcal{O}_\lambda$-prebasis is an $\mathcal{O}_\lambda$- basis iff the M_{∂_i}'s fulfill the integrability conditions.

Nota bene: ◇ $\partial_i : R_n/J \longrightarrow R_n/J$ is $\mathbb{C}$-linear, not $\mathbb{C}(x_1, \ldots, x_n)$-linear
◇ change of basis encoded by gauge transform

Writing integrable connections as cyclic R_n-modules

Input: Connection matrices $A_1, \ldots, A_n \in \mathbb{C}(x)^{m\times m}$ together with a choice of basis $\{f_1, \ldots, f_m\}$ of the $\mathbb{C}(x)$-vector space V.

Output: The ideal $J_{\mathcal{O}} \subset R_n$ such that $(V, \mathrm{d} - A\wedge) \cong R_n/J_{\mathcal{O}}$. in $\mathcal{O}$-border basis

Key assumption: One of $\{f_1, f_2, \ldots, f_m\}$ is a cyclic vector v of $(\mathbb{C}(x)^m, \nabla = \mathrm{d} - A\wedge)$.
Without loss of generality, $v = f_1$.

Procedure

1. Find an order ideal $\mathcal{O}$ of the form $\{\partial^{I_1}, \ldots, \partial^{I_m}\}$ with $I_1, \ldots, I_m \subset \mathbb{N}^n$ s.t. there exists an invertible matrix $g \in \mathbb{C}(x_1, \ldots, x_n)^{m\times m}$ for which
$$(\mathcal{O} \bullet f_1)^\top = g \cdot \begin{pmatrix} f_1 & f_2 & \cdots & f_m \end{pmatrix}^\top . \tag{5}$$
Such an $\mathcal{O}$ then yields a $\mathbb{C}(x)$-basis. The entries of g can be computed from $A_1, \ldots, A_n$.
2. Each $b_i \in \partial\mathcal{O}$ can be written as a multiple of one of the basis elements ∂^{I_k} by some ∂_j. The difference of b_i by the k-th entry of $\widetilde{A}_j \cdot (\partial^{I_1}, \ldots, \partial^{I_m})^\top$ yields an operator P_i that is marked by b_i.
3. Set $J_{\mathcal{O}}$ to be the R_n-ideal generated by P_i, $i = 1, \ldots, |\partial\mathcal{O}|$, arising from $\partial\mathcal{O}$.

Proposition: The operators P_i, $i = 1, \ldots, |\partial\mathcal{O}|$, as described above, are an $\mathcal{O}$-border basis of $J_{\mathcal{O}} \subset R_n$. In particular, $J_{\mathcal{O}}$ has holonomic rank m.

In our article: applied to PDEs behind integrals in theoretical physics.

Towards a classification of D_n-ideals

Can we use the theory of Hilb_n^m to study $\{I \subset D_n \mid \mathrm{rank}(J) = m\}$?

PDEs with constant coefficients: $I = D_n\mathfrak{J}$ with $\mathfrak{J} \subset \mathbb{C}[\partial_1, \ldots, \partial_n]$

- $\dim_{\mathbb{C}}(\mathbb{C}[\partial]/\mathfrak{J}) = \mathrm{rank}(I)$.
- Any $\mathbb{C}$-basis of $\mathbb{C}[\partial]/\mathfrak{J}$ is a $\mathbb{C}(x_1, \ldots, x_n)$-basis of R_n/R_nI.
- In these bases, the connection matrices have constant entries, and hence are pairwise commuting.

Frobenius ideals: $I = D_n\mathfrak{J}$ for $\mathfrak{J} \subset \mathbb{C}[\theta_1, \ldots, \theta_n]$.

- $\mathrm{rank}(I) = \dim_{\mathbb{C}}(\mathbb{C}[\theta]/\mathfrak{J})$.
- connection matrices: $\frac{1}{x_i} M_{\theta_i}^{\top}$.
- There exists a $\mathbb{C}(x_1, \ldots, x_n)$-basis of R_n/R_nI in which the A_i's are p.w. commuting.

Thanks for your attention!

Funding statement

The research of CR and ALS is funded by the European Union (ERC, UNIVERSE PLUS, 101118787). Views and opinions expressed are, however, those of the author(s) only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.

$V = \mathbb{C}(x_1, x_2)^3$ together with $\nabla = \mathrm{d}$ **trivial connection** $(n = 2, m = 3)$

in basis (e_1, e_2, e_3): $\nabla = \mathrm{d} - A\wedge$ $\quad A = 0$ the 3×3 zero matrix

$(V, \nabla) \cong (R_2/\langle \partial_1, \partial_2 \rangle)^3$ $\quad$ flat sections: $\mathrm{Const}(\mathbb{C}^2, \mathbb{C}^3)$

Aim:

1. Find $J \subset R_2$ such that $(V, \mathrm{d}) \cong R_2/J$ together with
2. a $\mathbb{C}(x_1, x_2)$-basis of R_s/J in which the Pfaffian system has only zero matrices.

Strategy: cyclic vector

1. For $v = (x_1^2, x_1, 1)$, the R_2-linear morphism determined by

$$\varphi\colon\ R_2/J \xrightarrow{\cong} \mathbb{C}(x_1, x_2)^3, \quad 1 \mapsto v$$

is an isomorphism of R_2-modules, where $J = \langle \partial_1^3, \partial_2 \rangle$ is holonomic of rank 3.

2. To determine a suitable $\mathbb{C}(x_1, x_2)$-basis (s_1, s_2, s_3) of R_2/J: determine preimages of the standard unit vectors under φ.

Setting $s_1 = \partial_1^2$, $s_2 := \partial_1 \cdot \left(1 - \frac{x_1^2}{2}\partial_1^2\right)$, and $s_3 := 1 - \frac{x_1^2}{2}\partial_1^2 - x_1 \cdot s_2$ results in $\frac{1}{2}s_1 \bullet v = e_1$, $s_2 \bullet v = e_2$ and $s_3 \bullet v = e_3$. In this basis, the Pfaffian system (A_1, A_2) indeed consists of the zero matrices only, since for any $f \in \mathrm{Sol}(J) = \mathbb{C} \cdot \{1, x_1, x_1^2\}$:

$$\partial_i \bullet \begin{pmatrix} s_1 \bullet f \\ s_2 \bullet f \\ s_3 \bullet f \end{pmatrix} = \partial_i \bullet \begin{pmatrix} \partial_1^2 \bullet f \\ -\frac{x_1^2}{2}\partial_1^3 \bullet f - x_1\partial_1^2 \bullet f + \partial_1 \bullet f \\ \frac{x_1^3}{2}\partial_1^3 \bullet f + \frac{x_1^2}{2}\partial_1^2 \bullet f - x_1\partial_1 \bullet f + f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad i = 1, 2.$$

From connection matrices to a border basis – an example

A border basis of $\boxed{I = \langle x\partial_x^2 - y\partial_y^2 + \partial_x - \partial_y, x\partial_x + \partial_y + 1\rangle \subset D_2.}$

- The Pfaffian system of I in the basis $\mathcal{O}_1 = \{1, \partial_x\}$ are

$$\boxed{A_1 = \begin{pmatrix} 0 & 1 \\ -\frac{1}{x(x-y)} & -\frac{3x-y}{x(x-y)} \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} -\frac{1}{y} & -\frac{x}{y} \\ \frac{1}{y(x-y)} & \frac{x+y}{y(x-y)} \end{pmatrix}.}$$

Computed via Gröbner basis methods in the rational Weyl algebra for weighted lexicographic order in [Section 3.1.4, 8].

- Via $\partial\mathcal{O}_1 = \{\partial_x^2, \partial_x\partial_y, \partial_y\}$, we associate the R_2-ideal

$$\boxed{J_{\mathcal{O}_1} = \left\langle \partial_x^2 + \frac{3x-y}{x(x-y)}\partial_x + \frac{1}{x(x-y)},\ \partial_x\partial_y + \frac{x+y}{y(y-x)}\partial_x + \frac{1}{y(y-x)},\ \partial_y + \frac{x}{y}\partial_x + \frac{1}{y}\right\rangle.}$$

- One has $R_n I = J_{\mathcal{O}_1}$. unique as R_2-ideal not unique as D_2-ideals

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[8] P. Görlach, J. Koefler, ALS, M. Sayrafi, H. Schroeder, N. Weiss, and F. Zaffalon. Connection Matrices in *Macaulay2*. Preprint arXiv:2504.01362.