

Encoding Holonomic D -Modules

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QM Research Seminar
Centre for Quantum Mathematics
Southern University of Denmark
September 17, 2026

Hilbert's 21st problem: *Given points $p_0, \dots, p_N$ in the complex plane and a representation $\rho : \pi_1(\mathbb{C} \setminus \{p_0, \dots, p_N\}) \rightarrow \mathrm{GL}(V)$ of the fundamental group, is there always a linear ODE of the Fuchsian class whose solutions realize these singularities and monodromy data?*

Answer: No! Bolibrukh

Riemann–Hilbert correspondence (Kashiwara, Mebkhout)

Let X be a smooth algebraic variety $_{\mathbb{C}}$ and $\mathcal{D}_X$ the sheaf of differential operators on X . The functor $\mathrm{Sol}(\bullet) = \mathcal{R}\mathrm{Hom}_{\mathcal{D}_X}(\bullet, \mathcal{O}_X^{\mathrm{an}})[\dim(X)]$ is an equivalence of categories

$$\mathrm{Sol} : \mathrm{D}_{\mathrm{rh}}^b(\mathcal{D}_X) \xrightarrow{\simeq} \mathrm{D}_c^b(\mathbb{C}_{X^{\mathrm{an}}}).$$

It restricts to $\mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_X) \xrightarrow{\simeq} \mathrm{Perv}(\mathbb{C}_{X^{\mathrm{an}}})$ and $\mathrm{Conn}^{\mathrm{reg}}(X; D) \xrightarrow{\simeq} \mathrm{Loc}(X^{\mathrm{an}} \setminus D).$

Today: How to encode and modify $\mathcal{D}$ -modules?

Some applications of $\mathcal{D}$ -modules: Hodge theory, singularity theory, homological mirror symmetry, GKZ systems, generating functions, scattering amplitudes, holonomic functions: numerical evaluation & symbolic computations, integral transforms, Stokes phenomena, ...

From sheaves of $\mathcal{D}$ -modules to D -modules

Fact: On D -affine schemes, $\Gamma(\bullet): \text{Mod}_{\text{qc}}(\mathcal{D}_X) \xrightarrow{\simeq} \text{Mod}(D_X(X)).$

A prominent example

For $X = \mathbb{A}_{\mathbb{C}}^n$, $D_X(X)$ is $D_n = \mathbb{C}[x_1, \dots, x_n] \langle \partial_1, \dots, \partial_n \rangle$, the (n -th) **Weyl algebra**.
obtained from the free algebra generated by $x_1, \dots, x_n, \partial_1, \dots, \partial_n$ by requiring that all variables commute except ∂_i, x_j : $[\partial_i, x_j] = \partial_i x_j - x_j \partial_i = 1$ (Leibniz' rule)

Elements of the Weyl algebra:

$$D_n = \left\{ \sum_{\alpha, \beta \in \mathbb{N}^n, \text{ finite}} c_{\alpha, \beta} x^{\alpha} \partial^{\beta} \mid c_{\alpha, \beta} \in \mathbb{C} \right\}$$

The left ideal $I = \langle P_1, \dots, P_k \rangle \subset D_n$ encodes the system $\{P_1 \bullet f = 0, \dots, P_k \bullet f = 0\}.$

More generally: left modules over D_n , $M \in \text{Mod}(D_n)$ $D_n \times M \rightarrow M, (P, m) \mapsto P \bullet m$

Examples:

◇ D_n/I for $I \subset D_n$ ◇ many function spaces

◇ M defined to make the sequence $D_n^k \xrightarrow{\varphi} D_n^{\ell} \rightarrow M \rightarrow 0$ exact, where

$$\varphi: (Q_1, \dots, Q_k) \mapsto \left(\sum_{i=1}^k Q_i P_{i1}, \dots, \sum_{i=1}^k Q_i P_{i\ell} \right) \quad \left\{ \sum_{j=1}^{\ell} P_{ij} \bullet f_j = 0, i = 1, \dots, k \right\}$$

Nota bene: Not every, but every *holonomic* D_n -module is of the form D_n/I .

$D_n = \mathbb{C}[x_1, \dots, x_n] \langle \partial_1, \dots, \partial_n \rangle$ (n -th) **Weyl algebra**

$R_n = \mathbb{C}(x_1, \dots, x_n) \langle \partial_1, \dots, \partial_n \rangle$ (n -th) **rational Weyl algebra**

Some definitions

- Let $(0, 1) = (0, \dots, 0, 1, \dots, 1) \in \mathbb{R}^{2n}$. The **initial form** of $P \in D$ is

$$\text{in}_{(0,1)}(P) = \sum_{\{(\alpha, \beta) | (\alpha, \beta) \cdot (0,1) \text{ maximal}\}} c_{\alpha, \beta} x^\alpha \partial^\beta.$$

The **characteristic ideal** of $I \subset D_n$ is

$$\text{in}_{(0,1)}(I) = \langle \text{in}_{(0,1)}(P) | P \in I \rangle \subset \mathbb{C}[x_1, \dots, x_n][\partial_1, \dots, \partial_n].$$

- The **singular locus** of $I \subset D_n$ is the variety of

$$(\text{in}_{(0,1)}(I) : \langle \partial_1, \dots, \partial_n \rangle^{(\infty)}) \cap \mathbb{C}[x_1, \dots, x_n].$$

- A D_n -ideal I is **holonomic** if $\dim V(\text{in}_{(0,1)}(I)) = n$. $\geq n$ by **Bernstein's inequality**
- The **holonomic rank** of a D_n -ideal I is the dimension of the $\mathbb{C}(x_1, \dots, x_n)$ -vector space underlying $R_n/R_n I \in \text{Mod}(R_n)$.

Fact: $I \subset D_n$ holonomic $\Rightarrow \text{rank}(I) < \infty$.

The reverse implication does not hold.

Holonomic functions

The **solutions** of a D_n -ideal I are (on $U \overset{\text{open}}{\subset} \mathbb{C}^n$)

$$\text{Sol}(I) := \text{Hom}_{D_n}(D_n/I, \mathcal{O}^{\text{an}}) \cong \{f \in \mathcal{O}^{\text{an}} \mid P \bullet f = 0 \text{ for all } P \in I\}.$$

A function $f(x_1, \dots, x_n)$ is **holonomic** if $f \in \text{Sol}(I)$ for some holonomic $I \subset D_n$.

Theorem (Cauchy–Kovalevskaya–Kashiwara)

Let $I \subset D_n$ be holonomic and $U \subset \mathbb{C}^n \setminus \text{Sing}(I)$ be a simply connected domain. Then the $\mathbb{C}$ -vector space of holomorphic solutions to I on U is finite-dimensional. More precisely,

$$\dim_{\mathbb{C}}(\text{Sol}(I)) = \text{rank}(I).$$

Some features of holonomic functions

- ◇ can be encoded by **finite data**
- ◇ numerical evaluation and minimization: holonomic gradient method and descent
- ◇ closure properties of the class of holonomic functions
 $+$, $\cdot$, $\int$, ∂_i , $*$, **holoalg**, $1/f$ iff f'/f is algebraic, **restrictions**, ...
- ◇ functorial operations in terms of $\mathcal{D}$ -modules

System of PDEs $\leftrightarrow$ 1st order matrix equations

An **integrable connection** is a pair (M, ∇) consisting of an m -dimensional $\mathbb{C}(x_1, \dots, x_n)$ -vector space M together with an **integrable** connection

$$\boxed{\nabla : M \longrightarrow M \otimes \Omega^1.} \quad (\mathbb{C}\text{-linear, Leibniz' rule, } \nabla^2 = 0)$$

Identifying $M \cong \mathbb{C}(x_1, \dots, x_n)^m$: $\nabla = d - A \wedge$, A an $m \times m$ matrix of differential 1-forms.
Geometrically: stalk of vector bundle on $\mathbb{A}_{\mathbb{C}}^n$ at the generic point.

Pfaffian system of D_n -ideals (in fact of R_n -ideals $J = R_n I$)

Let $I \subset D_n$ with $\text{rank}(I) = m$, $\{s_1 = 1, s_2, \dots, s_m\}$ a $\mathbb{C}(x_1, \dots, x_n)$ -basis of R_n/I .
There exist unique $A_1, \dots, A_n \in \text{Mat}_{m \times m}(\mathbb{C}(x_1, \dots, x_n))$ such that

$$\boxed{\partial_i \bullet (f, s_2 \bullet f, \dots, s_m \bullet f)^{\top} = A_i \cdot (f, s_2 \bullet f, \dots, s_m \bullet f)^{\top} \quad \text{for all } f \in \text{Sol}(I).}$$

The matrices A_i fulfill the **integrability conditions**: $[A_i, A_j] = \partial_i \bullet A_j - \partial_j \bullet A_i$ for all i, j .

Translating*: $\boxed{A = A_1 dx_1 + \dots + A_n dx_n}$ *duality swept under the rug

A D_n -module M is **cyclic** if there exists $v \in M$ such that $D_n \bullet v = M$.

Examples: D_n/I , holonomic D_n -modules $\supset$ integrable connections

Several choices to represent $I \subset D_n$:

- ◇ by arbitrary generators $I = \langle P_1, \dots, P_k \rangle$
- ◇ as Gröbner bases w.r.t. weight vectors or a chosen term order $\prec$
- ◇ via Stafford's theorem $I = \langle P_1, P_2 \rangle$

An A -hypergeometric example (a.k.a. GKZ system)

Let $I = H_A(\beta) \subset D_3$ for $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{pmatrix}$ and $\beta = (3 \ 4)^\top$. This GKZ system equals

$$\begin{aligned} H_A(\beta) &= \langle x_1 \partial_1 + 2x_2 \partial_2 + 3x_3 \partial_3 - 3, x_1 \partial_1 + x_3 \partial_3 - 4, \partial_1 \partial_2 - \partial_3 \rangle \\ &= \langle 2x_2 \partial_2 + 2x_3 \partial_3 + 1, x_1 \partial_1 + x_3 \partial_3 - 4, \partial_1 \partial_2 - \partial_3, \\ &\quad x_3 \partial_2 \partial_3 + x_1 \partial_3 - 4\partial_2, 2x_3 \partial_1 \partial_3 + 2x_2 \partial_3 + \partial_1, \\ &\quad 2x_3^2 \partial_3^2 - 2x_1 x_2 \partial_3 - 5x_3 \partial_3 - 4 \rangle. \end{aligned}$$

second: Gröbner basis w.r.t. $w = (0, 0, 0, 1, 1, 1)$

Today: Border bases for ideals in the rational Weyl algebra

Zero-dimensional ideals in polynomial rings

Let $S = \mathbb{C}[X_1, \dots, X_n]$. The **Hilbert scheme of m points in affine n -space**,

$$\text{Hilb}_n^m = \{I \subset S \mid \dim_{\mathbb{C}}(S/I) = m\},$$

classifies ideals $I \subset S$ whose quotient ring is m -dimensional as a $\mathbb{C}$ -vector space.

Properties of Hilb_n^m

- ◇ It is connected. [Hartshorne](#)
- ◇ It can be covered by open affine subschemes...
 - ▶ ...with a combinatorial description in terms of partitions.
 - ▶ ...whose ideals are naturally tied to [border bases](#)

Two questions

- ① Can this be transferred to the Weyl algebra?
- ② Is it useful in applications?

Answers in [3]: [Yes!](#)

[4] E. Miller and B. Sturmfels. *Combinatorial Comm. Algebra*, volume 227 of *Univ. Lecture Ser.*, Springer, 2005.

[5] M. Kreuzer and L. Robbiano. The geometry of border bases. *J. Pure Appl. Algebra*, 215:2005–2018, 2011.

[6] C. Rodriguez and ALS. Border Bases in the Rational Weyl Algebra. *Adv. Appl Math.*, 177:103065, 2026. [8/16](#)

A motivating example from the commutative case: 3 points in the plane

The Hilbert scheme Hilb_2^3 classifies $I \subset \mathbb{C}[X, Y]$ such that $\dim_{\mathbb{C}}(\mathbb{C}[X, Y]/I) = 3$.

- ◇ Hilb_2^3 can be covered by three open affine subschemes corresponding to monomial bases $(Y^2, Y, 1)$, $(X, Y, 1)$, and $(X^2, X, 1)$.
- ◇ Each $I \in \text{Hilb}_2^3$ in the second chart can be represented as

$$I = \left\langle \underline{X^2} - aX - bY - c, \underline{XY} - dX - eY - f, \underline{Y^2} - gX - hy - i \right\rangle. \quad (1)$$

- ◇ In the $\mathbb{C}$ -basis $(X, Y, 1)$ of $\mathbb{C}[X, Y]/I$, the maps $X \cdot, Y \cdot \in \text{End}_{\mathbb{C}}(\mathbb{C}[X, Y]/I)$ are

$$M_X = \begin{pmatrix} a & d & 1 \\ b & e & 0 \\ c & f & 0 \end{pmatrix} \quad \text{and} \quad M_Y = \begin{pmatrix} d & g & 0 \\ e & h & 1 \\ f & i & 0 \end{pmatrix}.$$

- ◇ In order to be a Gröbner basis, all S-pairs of the generators of I in (1) need to reduce to zero. This is equivalent to requiring that $M_X M_Y = M_Y M_X$.
- ◇ M_X and M_Y commute if and only if

$$f = bg - de, \quad c = -ae + bd - bh + e^2, \quad \text{and} \quad i = -ag + d^2 - dh + eg.$$

This yields a 6-dimensional affine open subscheme of Hilb_2^3 . denoted U_{2+1} in [1]

One ingredient of border bases: order ideals

An **order ideal** is a non-empty subset $\lambda \subset \mathbb{N}^n$ such that

$$u \in \lambda, v \in \mathbb{N}^n, v \leq u \implies v \in \lambda. \quad \leq \text{ component-wise}$$

Equivalently: $\mathcal{O}_\lambda = \{X^u | u \in \lambda\} \subset \mathbb{T}_n$ $\mathbb{T}_n$ the set of monomials in $X_1, \dots, X_n$

Example: For any choice of monomial order $\prec$, the set of **standard monomials** of a Gröbner basis of $I \subset S$ is an order ideal.
 $\{X^\alpha | X^\alpha \notin \text{in}_\prec(I)\}$

The **border** of an order ideal $\mathcal{O}_\lambda$ is

$$\partial \mathcal{O}_\lambda = (X_1 \mathcal{O}_\lambda \cup X_2 \mathcal{O}_\lambda \cup \dots \cup X_n \mathcal{O}_\lambda) \setminus \mathcal{O}_\lambda. \quad (2)$$

The **corners** of $\mathcal{O}_\lambda$ are the minimal gen's of the monomial ideal generated by $\mathbb{T}_n \setminus \mathcal{O}_\lambda$.

Example

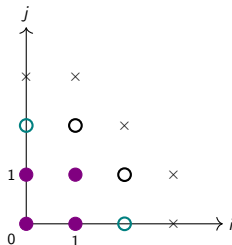
Let $\mathcal{O} = \{1, X_1, X_2, X_1 X_2\} \subset \mathbb{T}^2$.

• $\lambda = \{(0, 0), (1, 0), (0, 1), (1, 1)\} \subset \mathbb{N}^2$

◦ $\partial \mathcal{O} = \{X_1^2, X_1^2 X_2, X_1 X_2^2, X_2^2\}$

◦ corners of $\mathcal{O}$

× $\partial^2 \mathcal{O} = \{X_1^3, X_1^3 X_2, X_1^2 X_2^2, X_2^3 X_1, X_2^3\}$



If $\mathcal{O}_\lambda = \{t_1, t_2, \dots, t_m\} \subset \mathbb{T}_n$ is an order ideal and $\partial\mathcal{O}_\lambda = \{b_1, b_2, \dots, b_p\}$ its border, then an $\mathcal{O}_\lambda$ -**border prebasis** is a set of polynomials $G_\lambda = \{g_1, g_2, \dots, g_p\}$ where

$$\boxed{g_j = b_j - \sum_{i=1}^m c_{i,j} t_i} \quad \text{with } c_{i,j} \in \mathbb{C}. \quad g_j \text{ is "marked by" } b_j \quad (3)$$

Formal multiplication matrices associated to an $\mathcal{O}_\lambda$ -border prebasis

The k -th column of $M_{X_i} \in \text{Mat}_{m \times m}(\mathbb{C})$ is defined to be

$$\boxed{(M_{X_i})_{*k} = \begin{cases} e_r & \text{if } X_i t_k = t_r, \\ (c_{1,s}, \dots, c_{m,s})^\top & \text{if } X_i t_k = b_s, \end{cases}} \quad (4)$$

where e_r denotes the r -th standard unit vector in $\mathbb{R}^m$ and the $c_{j,s}$'s are as in (3).

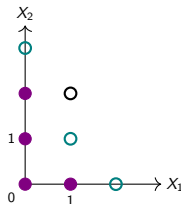
Terminology: The $(M_{X_i})_{*k}$ is the result of applying the division algorithm to $X_i t_k$.

$\mathcal{O}_\lambda = \{t_1, t_2, \dots, t_m\} \subset \mathbb{T}_n$ an order ideal

Definition: A border prebasis $G \subset I$ is an $\mathcal{O}_\lambda$ -**border basis** of $I \subset \mathbb{C}[X_1, \dots, X_n]$ if the residue classes of $t_1, t_2, \dots, t_m$ form a $\mathbb{C}$ -vector space basis of S/I .

Example: $I = \langle X_1^2 + X_2^2 - 2, X_1X_2 - 1 \rangle$

- ◇ $\dim_{\mathbb{C}}(\mathbb{C}[X_1, X_2]/I) = 4$ $(-1, -1), (1, 1)$ (double each)
- ◇ no border basis for $\{1, X_1, X_2, X_1X_2\}$
- ◇ $\mathcal{O}_\lambda = \{1, X_1, X_2, X_2^2\}$ $\partial\mathcal{O}_\lambda = \{X_1^2, X_1X_2, X_1X_2^2, X_2^3\}$
- ◇ $G = \{X_1^2 + X_2^2 - 2, X_1X_2 - 1, X_1X_2^2 - X_2, X_2^3 + X_1 - 2X_2\}$ is an $\mathcal{O}_\lambda$ -border basis



Properties

- ◇ Any border basis of I also generates I as an S -ideal.
- ◇ If it exists, an $\mathcal{O}_\lambda$ -border basis is unique.

Characterization

Let $\mathcal{O}_\lambda = \{t_1, t_2, \dots, t_m\}$ be an order ideal. Then an $\mathcal{O}_\lambda$ -border prebasis $G = \{g_1, \dots, g_p\}$ is an $\mathcal{O}_\lambda$ -border basis of $I = \langle g_1, \dots, g_p \rangle$ if and only if the formal multiplication matrices are pairwise commuting, i.e., if $M_{X_i}M_{X_j} = M_{X_j}M_{X_i}$ for all i, j .

Border bases in the rational Weyl algebra

Previously: $S = \mathbb{C}[X_1, \dots, X_n]$, $\mathbb{T}_n = \{X^\alpha \mid \alpha \in \mathbb{N}^n\}$ monomials in the X_i 's.

Now: $R_n = \mathbb{C}(x_1, \dots, x_n)\langle \partial_1, \dots, \partial_n \rangle$, $\mathbb{D}_n = \{\partial^\alpha \mid \alpha \in \mathbb{N}^n\}$ monomials in the ∂_i 's.
 $\mathbb{C}(x_1, \dots, x_n)$ as field of coefficients

Order ideals $\mathcal{O}_\lambda$, border prebases, formal multiplication matrices M_{∂_i} , border bases: definitions straightforward to transfer.

Proposition

Let $\mathcal{O}_\lambda = \{t_1, t_2, \dots, t_m\}$ be an order ideal, let $J \subset R_n$ with $\text{rank}(J) = m$. Assume that the residue classes of the t_i 's form a $\mathbb{C}(x_1, \dots, x_n)$ -vector space basis of R_n/J . Then:

- 1 There exists a unique $\mathcal{O}_\lambda$ -border basis G of J .
- 2 Let $G' \subset J$ be an $\mathcal{O}_\lambda$ -border prebasis. Then G' is the $\mathcal{O}_\lambda$ -border basis of J .

Characterization

$$[M_{\partial_i}, M_{\partial_j}] = \partial_j \bullet M_{\partial_i} - \partial_i \bullet M_{\partial_j}$$

An $\mathcal{O}_\lambda$ -prebasis is an $\mathcal{O}_\lambda$ -basis iff the M_{∂_i} 's fulfill the integrability conditions.

Nota bene: $\diamond \partial_i: R_n/J \rightarrow R_n/J$ is $\mathbb{C}$ -linear, not $\mathbb{C}(x_1, \dots, x_n)$ -linear
 $\diamond$ change of basis encoded by **gauge transform**

Writing integrable connections as cyclic R_n -modules

Input: Connection matrices $A_1, \dots, A_n \in \mathbb{C}(x)^{m \times m}$ together with a choice of basis $\{f_1, \dots, f_m\}$ of the $\mathbb{C}(x)$ -vector space V .

Output: The ideal $J_{\mathcal{O}} \subset R_n$ such that $(V, d - A\wedge) \cong R_n/J_{\mathcal{O}}$. **in $\mathcal{O}$ -border basis**

Key assumption: One of $\{f_1, f_2, \dots, f_m\}$ is a cyclic vector v of $(\mathbb{C}(x)^m, \nabla = d - A\wedge)$.

Without loss of generality, $v = f_1$.

Procedure

- 1 Find an order ideal $\mathcal{O}$ of the form $\{\partial^{l_1}, \dots, \partial^{l_m}\}$ with $l_1, \dots, l_m \subset \mathbb{N}^n$ s.t. there exists an invertible matrix $g \in \mathbb{C}(x_1, \dots, x_n)^{m \times m}$ for which

$$(\mathcal{O} \bullet f_1)^\top = g \cdot (f_1 \quad f_2 \quad \dots \quad f_m)^\top. \quad (5)$$

Such an $\mathcal{O}$ then yields a $\mathbb{C}(x)$ -basis. **The entries of g can be computed from $A_1, \dots, A_n$.**

- 2 Each $b_i \in \partial\mathcal{O}$ can be written as a multiple of one of the basis elements ∂^{l_k} by some ∂_j . The difference of b_i by the k -th entry of $\tilde{A}_j \cdot (\partial^{l_1}, \dots, \partial^{l_m})^\top$ yields an operator P_i that is marked by b_i . $\tilde{A}_j = gA_jg^{-1} + \partial g/\partial x_j g^{-1}$
- 3 Set $J_{\mathcal{O}}$ to be the R_n -ideal generated by P_i , $i = 1, \dots, |\partial\mathcal{O}|$, arising from $\partial\mathcal{O}$.

Proposition: The operators P_i , $i = 1, \dots, |\partial\mathcal{O}|$, as described above, are an $\mathcal{O}$ -border basis of $J_{\mathcal{O}} \subset R_n$. In particular, $J_{\mathcal{O}}$ has holonomic rank m .

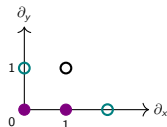
In our article: applied to PDEs behind integrals in theoretical physics.

From connection matrices to a border basis – a small example

Let $I = \langle x\partial_x^2 - y\partial_y^2 + \partial_x - \partial_y, x\partial_x + y\partial_y + 1 \rangle \subset D_2.$ $\text{rank}(I) = 2$

- ◇ The Pfaffian system of I in the basis $\mathcal{O}_1 = \{1, \partial_x\}$ is given by

$$A_1 = \begin{pmatrix} 0 & 1 \\ -\frac{1}{x(x-y)} & -\frac{3x-y}{x(x-y)} \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} -\frac{1}{y} & -\frac{x}{y} \\ \frac{1}{y(x-y)} & \frac{x+y}{y(x-y)} \end{pmatrix}.$$



Computed via Gröbner basis methods in the rational Weyl algebra for weighted lexicographic order in [Section 3.1.4, 8].

- ◇ Via $\partial\mathcal{O}_1 = \{\partial_x^2, \partial_x\partial_y, \partial_y\}$, we associate the R_2 -ideal

$$J_{\mathcal{O}_1} = \left\langle \partial_x^2 + \frac{3x-y}{x(x-y)}\partial_x + \frac{1}{x(x-y)}, \partial_x\partial_y + \frac{x+y}{y(y-x)}\partial_x + \frac{1}{y(y-x)}, \partial_y + \frac{x}{y}\partial_x + \frac{1}{y} \right\rangle,$$

given as a border basis.

- ◇ One has $R_n I = J_{\mathcal{O}_1}$. unique as R_2 -ideal (in general) not unique as D_2 -ideals

Towards a classification of D_n -ideals

Can we use the theory of Hilb_n^m to study $\{I \subset D_n \mid \text{rank}(I) = m\}$?

PDEs with constant coefficients: $I = D_n \mathfrak{J}$ with $\mathfrak{J} \subset \mathbb{C}[\partial_1, \dots, \partial_n]$

- ◇ $\dim_{\mathbb{C}}(\mathbb{C}[\partial]/\mathfrak{J}) = \text{rank}(I)$.
- ◇ Any $\mathbb{C}$ -basis of $\mathbb{C}[\partial]/\mathfrak{J}$ is a $\mathbb{C}(x_1, \dots, x_n)$ -basis of $R_n/R_n I$.
- ◇ In these bases, the connection matrices have constant entries, and hence are pairwise commuting.

Frobenius ideals: $I = D_n \mathfrak{J}$ for $\mathfrak{J} \subset \mathbb{C}[\theta_1, \dots, \theta_n]$. $\theta_i = x_i \partial_i$

- ◇ $\text{rank}(I) = \dim_{\mathbb{C}}(\mathbb{C}[\theta]/\mathfrak{J})$.
- ◇ connection matrices: $\frac{1}{x_i} M_{\theta_i}^{\top}$.
- ◇ There exists a $\mathbb{C}(x_1, \dots, x_n)$ -basis of $R_n/R_n I$ in which the A_i 's are p.w. commuting.

Tak for jeres opmærksomhed!

Funding statement

The research of ALS is funded by the European Union (ERC, UNIVERSE PLUS, 101118787). Views and opinions expressed are, however, those of the author(s) only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.

Carrying bases along [Example 1.2, 1]

$V = \mathbb{C}(x_1, x_2)^3$ together with $\nabla = d$ **trivial connection** ($n = 2, m = 3$)
in basis (e_1, e_2, e_3) : $\nabla = d - A \wedge$ $A = 0$ the 3×3 zero matrix
 $(V, \nabla) \cong (R_2 / \langle \partial_1, \partial_2 \rangle)^3$ flat sections: $\text{Const}(\mathbb{C}^2, \mathbb{C}^3)$

Aim:

- Find $J \subset R_2$ such that $(V, d) \cong R_2/J$ together with
- a $\mathbb{C}(x_1, x_2)$ -basis of R_2/J in which the Pfaffian system has only zero matrices.

Strategy: cyclic vector

- For $v = (x_1^2, x_1, 1)$, the R_2 -linear morphism determined by

$$\varphi: R_2/J \xrightarrow{\cong} \mathbb{C}(x_1, x_2)^3, \quad 1 \mapsto v$$

is an isomorphism of R_2 -modules, where $J = \langle \partial_1^3, \partial_2 \rangle$ is holonomic of rank 3.

- To determine a suitable $\mathbb{C}(x_1, x_2)$ -basis (s_1, s_2, s_3) of R_2/J : determine preimages of the standard unit vectors under φ .

Setting $s_1 = \partial_1^2$, $s_2 := \partial_1 \cdot (1 - \frac{x_1^2}{2} \partial_1^2)$, and $s_3 := 1 - \frac{x_1^2}{2} \partial_1^2 - x_1 \cdot s_2$ results in $\frac{1}{2}s_1 \bullet v = e_1$, $s_2 \bullet v = e_2$ and $s_3 \bullet v = e_3$. In this basis, the Pfaffian system (A_1, A_2) indeed consists of the zero matrices only, since for any $f \in \text{Sol}(J) = \mathbb{C} \cdot \{1, x_1, x_1^2\}$:

$$\partial_i \bullet \begin{pmatrix} s_1 \bullet f \\ s_2 \bullet f \\ s_3 \bullet f \end{pmatrix} = \partial_i \bullet \begin{pmatrix} \partial_1^2 \bullet f \\ -\frac{x_1^2}{2} \partial_1^3 \bullet f - x_1 \partial_1^2 \bullet f + \partial_1 \bullet f \\ \frac{x_1^3}{2} \partial_1^3 \bullet f + \frac{x_1^2}{2} \partial_1^2 \bullet f - x_1 \partial_1 \bullet f + f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad i = 1, 2.$$