

Border Bases in the Rational Weyl Algebra

Based on joint work with Carlos Rodriguez.

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Zero-dimensional ideals in polynomial rings

Let $S = \mathbb{C}[X_1, \dots, X_n]$. The **Hilbert scheme of m points in affine n -space**,

$$\text{Hilb}_n^m = \{I \subset S \mid \dim_{\mathbb{C}}(S/I) = m\},$$

classifies ideals $I \subset S$ whose quotient ring is m -dimensional as a $\mathbb{C}$ -vector space.

Properties of Hilb_n^m

- ◇ It is connected. [Hartshorne](#)
- ◇ It can be covered by open affine subschemes. . .
 - ▶ ... with a combinatorial description in terms of partitions of integers.
 - ▶ ... whose ideals are naturally tied to [border bases](#)

Two questions

- ① Can this be transferred to the Weyl algebra?
- ② Is it useful in applications?

Answers in [3]: **Yes!**

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- [1] E. Miller and B. Sturmfels. *Combinatorial Comm. Algebra*, volume 227 of *Univ. Lecture Ser.*, Springer, 2005.
[2] M. Kreuzer and L. Robbiano. The geometry of border bases. *J. Pure Appl. Algebra*, 215:2005–2018, 2011.
[3] C. Rodriguez and ALS. Border Bases in the Rational Weyl Algebra. *Adv. Appl Math.*, 177:103065, 2026. [2/14](#)

A motivating example from the commutative case: 3 points in the plane

The Hilbert scheme Hilb_2^3 classifies $I \subset \mathbb{C}[X, Y]$ such that $\dim_{\mathbb{C}}(\mathbb{C}[X, Y]/I) = 3$.

- ◇ Hilb_2^3 can be covered by three open affine subschemes corresponding to monomial bases $(Y^2, Y, 1)$, $(X, Y, 1)$, and $(X^2, X, 1)$.
- ◇ Each $I \in \text{Hilb}_2^3$ in the second chart can be represented as

$$I = \left\langle \underline{X^2} - aX - bY - c, \underline{XY} - dX - eY - f, \underline{Y^2} - gX - hy - i \right\rangle. \quad (1)$$

- ◇ In the $\mathbb{C}$ -basis $(X, Y, 1)$ of $\mathbb{C}[X, Y]/I$, the maps $X \cdot, Y \cdot \in \text{End}_{\mathbb{C}}(\mathbb{C}[X, Y]/I)$ are

$$M_X = \begin{pmatrix} a & d & 1 \\ b & e & 0 \\ c & f & 0 \end{pmatrix} \quad \text{and} \quad M_Y = \begin{pmatrix} d & g & 0 \\ e & h & 1 \\ f & i & 0 \end{pmatrix}.$$

- ◇ In order to be a Gröbner basis, all S-pairs of the generators of I in (1) need to reduce to zero. This is equivalent to requiring that $M_X M_Y = M_Y M_X$.
- ◇ M_X and M_Y commute if and only if

$$f = bg - de, \quad c = -ae + bd - bh + e^2, \quad \text{and} \quad i = -ag + d^2 - dh + eg.$$

This yields a 6-dimensional affine open subscheme of Hilb_2^3 . denoted U_{2+1} in [1]

One ingredient of border bases: order ideals

An **order ideal** is a non-empty subset $\lambda \subset \mathbb{N}^n$ such that

$$u \in \lambda, v \in \mathbb{N}^n, v \leq u \implies v \in \lambda. \quad \leq \text{ component-wise}$$

Equivalently: $\mathcal{O}_\lambda = \{X^u | u \in \lambda\} \subset \mathbb{T}_n$ $\mathbb{T}_n$ the set of monomials in $X_1, \dots, X_n$

Example: For any choice of monomial order $\prec$, the set of **standard monomials** of a Gröbner basis of $I \subset S$ is an order ideal.
 $\{X^\alpha | X^\alpha \notin \text{in}_\prec(I)\}$

The **border** of an order ideal $\mathcal{O}_\lambda$ is

$$\partial \mathcal{O}_\lambda = (X_1 \mathcal{O}_\lambda \cup X_2 \mathcal{O}_\lambda \cup \dots \cup X_n \mathcal{O}_\lambda) \setminus \mathcal{O}_\lambda. \quad (2)$$

The **corners** of $\mathcal{O}_\lambda$ are the minimal gen's of the monomial ideal generated by $\mathbb{T}_n \setminus \mathcal{O}_\lambda$.

Example

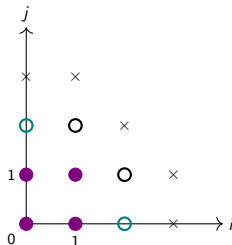
Let $\mathcal{O} = \{1, X_1, X_2, X_1 X_2\} \subset \mathbb{T}^2$.

• $\lambda = \{(0, 0), (1, 0), (0, 1), (1, 1)\} \subset \mathbb{N}^2$

◦ $\partial \mathcal{O} = \{X_1^2, X_1^2 X_2, X_1 X_2^2, X_2^2\}$

◦ corners of $\mathcal{O}$

× $\partial^2 \mathcal{O} = \{X_1^3, X_1^3 X_2, X_1^2 X_2^2, X_2^3 X_1, X_2^3\}$



If $\mathcal{O}_\lambda = \{t_1, t_2, \dots, t_m\} \subset \mathbb{T}_n$ is an order ideal and $\partial\mathcal{O}_\lambda = \{b_1, b_2, \dots, b_p\}$ its border, then an $\mathcal{O}_\lambda$ -**border prebasis** is a set of polynomials $G_\lambda = \{g_1, g_2, \dots, g_p\}$ where

$$\boxed{g_j = b_j - \sum_{i=1}^m c_{i,j} t_i} \quad \text{with } c_{i,j} \in \mathbb{C}. \quad g_j \text{ is "marked by" } b_j \quad (3)$$

Formal multiplication matrices associated to an $\mathcal{O}_\lambda$ -border prebasis

The k -th column of $M_{X_i} \in \text{Mat}_{m \times m}(\mathbb{C})$ is defined to be

$$\boxed{(M_{X_i})_{*k} = \begin{cases} e_r & \text{if } X_i t_k = t_r, \\ (c_{1,s}, \dots, c_{m,s})^\top & \text{if } X_i t_k = b_s, \end{cases}} \quad (4)$$

where e_r denotes the r -th standard unit vector in $\mathbb{R}^m$ and the $c_{j,s}$'s are as in (3).

Terminology: The $(M_{X_i})_{*k}$ is the result of applying the division algorithm to $X_i t_k$.

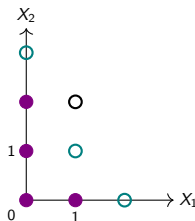
Example

$$I = \langle X_1^2 + X_2^2 - 2, X_1X_2 - 1 \rangle \subset \mathbb{C}[X_1, X_2]$$

- ◇ $\dim_{\mathbb{C}}(\mathbb{C}[X_1, X_2]/I) = 4$ $(-1, -1), (1, 1)$ (double each)
- ◇ $\mathcal{O}_{\lambda} = \{1, X_1, X_2, X_2^2\}$, $\partial\mathcal{O}_{\lambda} = \{X_1^2, X_1X_2, X_1X_2^2, X_2^3\}$
- ◇ $G = \{X_1^2 + X_2^2 - 2, X_1X_2 - 1, X_1X_2^2 - X_2, X_2^3 + X_1 - 2X_2\}$ is an $\mathcal{O}_{\lambda}$ -border prebasis
- ◇ formal multiplication matrices of G :

$$M_{X_1} = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad M_{X_2} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

- ◇ observation: $[M_{X_1}, M_{X_2}] = 0$



$I \subset S = \mathbb{C}[X_1, \dots, X_n]$	a zero-dimensional ideal
$\mathcal{O}_\lambda = \{t_1, t_2, \dots, t_m\} \subset \mathbb{T}_n$	an order ideal
$G = \{g_1, g_2, \dots, g_p\}$	an $\mathcal{O}_\lambda$ -border prebasis

Definition

A border prebasis G is an $\mathcal{O}_\lambda$ -**border basis** of I if the residue classes of $t_1, t_2, \dots, t_m$ form a $\mathbb{C}$ -vector space basis of S/I .

Properties

- ◇ Any border basis of I also generates I as an S -ideal.
- ◇ If it exists, an $\mathcal{O}_\lambda$ -border basis is unique.

Characterization

Let $\mathcal{O}_\lambda = \{t_1, t_2, \dots, t_m\}$ be an order ideal. Then an $\mathcal{O}_\lambda$ -border prebasis $G = \{g_1, \dots, g_p\}$ is an $\mathcal{O}_\lambda$ -border basis of $I = \langle g_1, \dots, g_p \rangle$ if and only if the formal multiplication matrices are pairwise commuting, i.e., if $M_{X_i} M_{X_j} = M_{X_j} M_{X_i}$ for all i, j .

Example cont'd: $I = \langle X_1^2 + X_2^2 - 2, X_1 X_2 - 1 \rangle$

- ◇ For $\mathcal{O}_\lambda = \{1, X_1, X_2, X_2^2\}$, G as before is an $\mathcal{O}_\lambda$ -border basis.
- ◇ No border basis exists for $\{1, X_1, X_2, X_1 X_2\}$.

Holonomic D -modules

$$\begin{array}{lll} D_n = \mathbb{C}[x_1, \dots, x_n] \langle \partial_1, \dots, \partial_n \rangle & (n\text{-th}) \text{ Weyl algebra} & [\partial_i, x_j] = \delta_{ij} \\ R_n = \mathbb{C}(x_1, \dots, x_n) \langle \partial_1, \dots, \partial_n \rangle & (n\text{-th}) \text{ rational Weyl algebra} & \\ I \subset D_n, J \subset R_n & \text{left ideals} & D_n/I \in \text{Mod}(D_n) \end{array}$$

Notation: $\bullet : D_n \times M \longrightarrow M, (P, m) \mapsto P \bullet m$ E.g., $\partial_i \bullet f = \frac{\partial f}{\partial x_i}$.

The **holonomic rank** of a D_n -ideal I is the dimension of the $\mathbb{C}(x_1, \dots, x_n)$ -vector space underlying $R_n/R_n I \in \text{Mod}(R_n)$.

Elements of the Weyl algebra:

$$D_n = \left\{ \sum_{\alpha, \beta \in \mathbb{N}^n, \text{ finite}} c_{\alpha, \beta} x^\alpha \partial^\beta \mid c_{\alpha, \beta} \in \mathbb{C} \right\}$$

Principal symbols

Let $(0, 1) = (0, \dots, 0, 1, \dots, 1) \in \mathbb{R}^{2n}$. The **initial form** of $P \in D$ is

$$\text{in}_{(0,1)}(P) = \sum_{\{(\alpha, \beta) \mid (\alpha, \beta) \cdot (0, 1) \text{ maximal}\}} c_{\alpha, \beta} x^\alpha \partial^\beta.$$

A D_n -ideal I is **holonomic** if $\dim V(\text{in}_{(0,1)}(I)) = n$. $\geq n$ by Bernstein's inequality

Fact: $I \subset D_n$ holonomic $\Rightarrow \text{rank}(I) < \infty$. The reverse implication does not hold.

System of higher-order PDEs $\leftrightarrow$ 1st order matrix equations

An **integrable connection** is a pair (M, ∇) consisting of an m -dimensional $\mathbb{C}(x_1, \dots, x_n)$ -vector space M together with an **integrable** connection

$$\boxed{\nabla : M \longrightarrow M \otimes \Omega^1.} \quad (\mathbb{C}\text{-linear, Leibniz' rule, } \nabla^2 = 0)$$

Identifying $M \cong \mathbb{C}(x_1, \dots, x_n)^m$: $\nabla = d - A \wedge$, A an $m \times m$ matrix of differential 1-forms.
Geometrically: stalk of vector bundle on $\mathbb{A}_{\mathbb{C}}^n$ at the generic point.

Pfaffian system of D_n -ideals (in fact of R_n -ideals $J = R_n I$)

Let $I \subset D_n$ with $\text{rank}(I) = m$, $\{s_1 = 1, s_2, \dots, s_m\}$ a $\mathbb{C}(x_1, \dots, x_n)$ -basis of R_n/I .
There exist unique $A_1, \dots, A_n \in \text{Mat}_{m \times m}(\mathbb{C}(x_1, \dots, x_n))$ such that

$$\boxed{\partial_i \bullet (f, s_2 \bullet f, \dots, s_m \bullet f)^{\top} = A_i \cdot (f, s_2 \bullet f, \dots, s_m \bullet f)^{\top} \quad \text{for all } f \in \text{Sol}(I).}$$

The matrices A_i fulfill the **integrability conditions**: $[A_i, A_j] = \partial_i \bullet A_j - \partial_j \bullet A_i$ for all i, j .

Translating*: $\boxed{A = A_1 dx_1 + \dots + A_n dx_n}$ * duality swept under the rug

$M \in \text{Mod}(D_n)$ is **cyclic** if there exists $v \in M$ such that $D_n \bullet v = M$.

Examples: D_n/I , holonomic D_n -modules $\supset$ integrable connections

[6] R. Hotta, K. Takeuchi, and T. Tanisaki. *D-Modules, Perverse Sheaves, and Representation Theory*, volume 236 of *Progress in Mathematics*. Birkhäuser, 2008.

[7] J. T. Stafford. Module structure of Weyl algebras. *J. London Math. Soc.*, 18:429–442, 1978.

Several choices to represent $I \subset D_n$:

- ◇ by arbitrary generators $I = \langle P_1, \dots, P_k \rangle$
- ◇ as Gröbner bases w.r.t. weight vectors or a chosen term order $\prec$
- ◇ via Stafford's theorem $I = \langle P_1, P_2 \rangle$

An A -hypergeometric example (a.k.a. GKZ system)

Let $I = H_A(\beta) \subset D_3$ for $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{pmatrix}$ and $\beta = (3 \ 4)^\top$. This GKZ system equals

$$\begin{aligned} H_A(\beta) &= \langle x_1 \partial_1 + 2x_2 \partial_2 + 3x_3 \partial_3 - 3, x_1 \partial_1 + x_3 \partial_3 - 4, \partial_1 \partial_2 - \partial_3 \rangle \\ &= \langle 2x_2 \partial_2 + 2x_3 \partial_3 + 1, x_1 \partial_1 + x_3 \partial_3 - 4, \partial_1 \partial_2 - \partial_3, \\ &\quad x_3 \partial_2 \partial_3 + x_1 \partial_3 - 4\partial_2, 2x_3 \partial_1 \partial_3 + 2x_2 \partial_3 + \partial_1, \\ &\quad 2x_3^2 \partial_3^2 - 2x_1 x_2 \partial_3 - 5x_3 \partial_3 - 4 \rangle. \end{aligned}$$

second: Gröbner basis w.r.t. $w = (0, 0, 0, 1, 1, 1)$

Today: Border bases for ideals in the rational Weyl algebra

Border bases in the rational Weyl algebra

Previously: $S = \mathbb{C}[X_1, \dots, X_n]$, $\mathbb{T}_n = \{X^\alpha \mid \alpha \in \mathbb{N}^n\}$ monomials in the X_i 's.

Now: $R_n = \mathbb{C}(x_1, \dots, x_n)\langle \partial_1, \dots, \partial_n \rangle$, $\mathbb{D}_n = \{\partial^\alpha \mid \alpha \in \mathbb{N}^n\}$ monomials in the ∂_i 's.
 $\mathbb{C}(x_1, \dots, x_n)$ as field of coefficients

Order ideals $\mathcal{O}_\lambda$, border prebases, formal multiplication matrices M_{∂_i} , border bases: definitions straightforward to transfer.

Proposition

Let $\mathcal{O}_\lambda = \{t_1, t_2, \dots, t_m\}$ be an order ideal, let $J \subset R_n$ with $\text{rank}(J) = m$. Assume that the residue classes of the t_i 's form a $\mathbb{C}(x_1, \dots, x_n)$ -vector space basis of R_n/J . Then:

- 1 There exists a unique $\mathcal{O}_\lambda$ -border basis G of J .
- 2 Let $G' \subset J$ be an $\mathcal{O}_\lambda$ -border prebasis. Then G' is the $\mathcal{O}_\lambda$ -border basis of J .

Characterization

$$[M_{\partial_i}, M_{\partial_j}] = \partial_j \bullet M_{\partial_i} - \partial_i \bullet M_{\partial_j}$$

An $\mathcal{O}_\lambda$ -prebasis is an $\mathcal{O}_\lambda$ -basis iff the M_{∂_i} 's fulfill the integrability conditions.

Nota bene: $\diamond \partial_i: R_n/J \rightarrow R_n/J$ is $\mathbb{C}$ -linear, not $\mathbb{C}(x_1, \dots, x_n)$ -linear
 $\diamond$ change of basis encoded by gauge transformation for connection matrices

Writing integrable connections as cyclic R_n -modules

Input: Connection matrices $A_1, \dots, A_n \in \mathbb{C}(x)^{m \times m}$ together with a choice of basis $\{f_1, \dots, f_m\}$ of the $\mathbb{C}(x)$ -vector space V .

Output: The ideal $J_{\mathcal{O}} \subset R_n$ such that $(V, d - A \wedge) \cong R_n/J_{\mathcal{O}}$. **in $\mathcal{O}$ -border basis**

Key assumption: One of $\{f_1, f_2, \dots, f_m\}$ is a cyclic vector v of $(\mathbb{C}(x)^m, \nabla = d - A \wedge)$.

Without loss of generality, $v = f_1$.

Procedure

- 1 Find an order ideal $\mathcal{O}$ of the form $\{\partial^{l_1}, \dots, \partial^{l_m}\}$ with $l_1, \dots, l_m \subset \mathbb{N}^n$ s.t. there exists an invertible matrix $g \in \mathbb{C}(x_1, \dots, x_n)^{m \times m}$ for which

$$(\mathcal{O} \bullet f_1)^\top = g \cdot (f_1 \quad f_2 \quad \dots \quad f_m)^\top. \quad (5)$$

Such an $\mathcal{O}$ then yields a $\mathbb{C}(x)$ -basis. **The entries of g can be computed from $A_1, \dots, A_n$.**

- 2 Each $b_i \in \partial\mathcal{O}$ can be written as a multiple of one of the basis elements ∂^{l_k} by some ∂_j . The difference of b_i by the k -th entry of $\tilde{A}_j \cdot (\partial^{l_1}, \dots, \partial^{l_m})^\top$ yields an operator P_i that is marked by b_i . $\tilde{A}_j = gA_jg^{-1} + \partial g/\partial x_j g^{-1}$
- 3 Set $J_{\mathcal{O}}$ to be the R_n -ideal generated by P_i , $i = 1, \dots, |\partial\mathcal{O}|$, arising from $\partial\mathcal{O}$.

Proposition: The operators P_i , $i = 1, \dots, |\partial\mathcal{O}|$, as described above, are an $\mathcal{O}$ -border basis of $J_{\mathcal{O}} \subset R_n$. In particular, $J_{\mathcal{O}}$ has holonomic rank m .

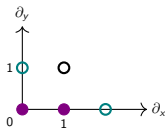
In our article: applied to PDEs behind integrals in theoretical physics.

From connection matrices to a border basis – an example

Let $I = \langle x\partial_x^2 - y\partial_y^2 + \partial_x - \partial_y, x\partial_x + y\partial_y + 1 \rangle \subset D_2$. $\text{rank}(I) = 2$

- ◇ The Pfaffian system of I in the basis $\mathcal{O}_1 = \{1, \partial_x\}$ is given by

$$A_1 = \begin{pmatrix} 0 & 1 \\ -\frac{1}{x(x-y)} & -\frac{3x-y}{x(x-y)} \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} -\frac{1}{y} & -\frac{x}{y} \\ \frac{1}{y(x-y)} & \frac{x+y}{y(x-y)} \end{pmatrix}.$$



Computed via Gröbner basis methods in the rational Weyl algebra for weighted lexicographic order in [Section 3.1.4, 8].

- ◇ Via $\partial\mathcal{O}_1 = \{\partial_x^2, \partial_x\partial_y, \partial_y\}$, we associate the R_2 -ideal

$$J_{\mathcal{O}_1} = \left\langle \partial_x^2 + \frac{3x-y}{x(x-y)}\partial_x + \frac{1}{x(x-y)}, \partial_x\partial_y + \frac{x+y}{y(y-x)}\partial_x + \frac{1}{y(y-x)}, \partial_y + \frac{x}{y}\partial_x + \frac{1}{y} \right\rangle,$$

given as a border basis.

- ◇ One has $R_n I = J_{\mathcal{O}_1}$. unique as R_2 -ideal (in general) not unique as D_2 -ideals

Problem/challenge: No implementation available yet!

Towards a classification of D_n -ideals

Can we use the theory of Hilb_n^m to study $\{I \subset D_n \mid \text{rank}(J) = m\}$?

PDEs with constant coefficients: $I = D_n \mathfrak{J}$ with $\mathfrak{J} \subset \mathbb{C}[\partial_1, \dots, \partial_n]$

- ◇ $\dim_{\mathbb{C}}(\mathbb{C}[\partial]/\mathfrak{J}) = \text{rank}(I)$.
- ◇ Any $\mathbb{C}$ -basis of $\mathbb{C}[\partial]/\mathfrak{J}$ is a $\mathbb{C}(x_1, \dots, x_n)$ -basis of $R_n/R_n I$.
- ◇ In these bases, the connection matrices have constant entries, and hence are pairwise commuting.

Frobenius ideals: $I = D_n \mathfrak{J}$ for $\mathfrak{J} \subset \mathbb{C}[\theta_1, \dots, \theta_n]$.

- ◇ $\text{rank}(I) = \dim_{\mathbb{C}}(\mathbb{C}[\theta]/\mathfrak{J})$.
- ◇ connection matrices: $\frac{1}{x_i} M_{\theta_i}^{\top}$.
- ◇ There exists a $\mathbb{C}(x_1, \dots, x_n)$ -basis of $R_n/R_n I$ in which the A_i 's are p.w. commuting.

Thanks for your attention!

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